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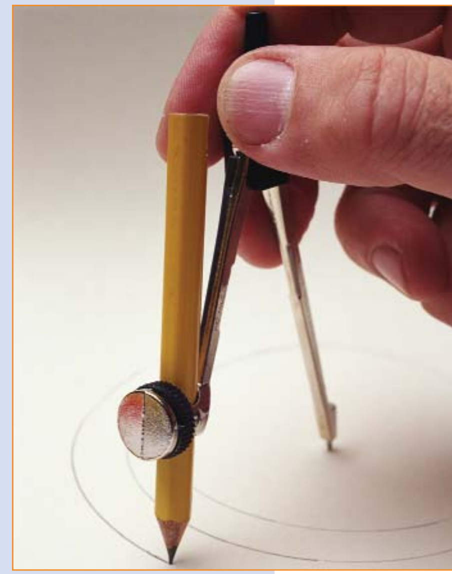
PREFACE

To the Teacher,

Analytic Analysis of Lines and Circles is designed to provide students with the analytic geometry background needed for further college-level geometry courses. Analytic geometry can be defined as algebraic analysis applied to geometrical concepts and figures, or the use of geometrical concepts and figures to illustrate algebraic forms.



Analytic geometry has many applications in different branches of science and makes it easier to solve a wide variety of problems. The goal of this text is to help students develop the skills necessary for solving analytic geometry problems, and then help students apply these skills. By the end of the book, students will have a good understanding of the analytic approach to solving problems. In addition, we have provided many systematic explanations throughout the text that will help instructors to reach the goals that they have set for their students. As always, we have taken particular care to create a book that students can read, understand, and enjoy, and that will help students gain confidence in their ability to use analytic geometry.

To the Student,

This book consists of two chapters, which cover analytical analysis of lines and circles respectively. Each chapter begins with basic definitions, theorems, and explanations which are necessary for understanding the subsequent chapter material. In addition, each chapter is divided into subsections so that students can follow the material easily.

Every subsection includes self-test **Check Yourself** problem sections followed by basic examples illustrating the relevant definition, theorem, rule, or property. Teachers should encourage their students to solve Check Yourself problems themselves because these problems are fundemental to understanding and learning the related subjects or sections. The answers to most Check Yourself problems are given directly after the problems, so that students have immediate feedback on their progress. Answers to some Check Yourself problems are not included in the answer key, as they are basic problems which are covered in detail in the preceding text or examples.

Giving answers to such problems would effectively make the problems redundant, so we have chosen to omit them, and leave students to find the basic answers themselves.

At the end of every section there are exercises categorized according to the structure and subject matter of the section. **Exercises** are graded in order, from easy (at the beginning) to difficult (at the end). Exercises which involve more ability and effort are denoted by one or two stars. In addition, exercises which deal with more than one subject are included in a separate bank of mixed problems at the end of the section. This organization allows the instructor to deal with only part of a section if necessary and to easily determine which exercises are appropriate to assign.

EXERCISES 1.1

A. Analytic Analysis of Points

1. Plot the following points in the coordinate pl

a. $A(2, 3)$ b. $B(-3, 1)$ c. $C(-3, 2)$

d. $D(5, -3)$ e. $E(0, -4)$ f. $F(-3, 0)$

Every chapter ends with three important sections.

The **Chapter Summary** is a list of important concepts and formulas covered in the chapter that students can use easily to get direct information whenever needed.

CHAPTER SUMMARY

- There is a one-to-one correspondence between th in a plane and the Cartesian coordinates. The po be represented by two components, the abscissa ordinate, $A(x, y)$.

A **Concept Check** section contains questions about the main concepts of the subjects covered, especially about the definitions, theorems or derived formulas.

Concept Check

1. What is the coordinate plane?
2. How can a point be represented in the coordinate
3. Define the concept of line. Find examples from

Finally, a **Chapter Review Test** section consists of three tests, each with sixteen carefully-selected problems. The first test covers primitive and basic problems. The second and third tests include more complex problems. These tests help students assess their ability in understanding the coverage of the chapter.

CHAPTER REVIEW TEST 1A

1. What is the length of the median passing throu the vertex A of a triangle ABC with vertices $A(4, 7$ $B(-1, 2)$, and $C(3, 4)$?

A) 5 B) 6 C) 7 D) 8 E)

The answers to the exercises and the tests are given at the end of the book so that students can compare their solution with the correct answer.

Each chapter also includes some subjects which are denoted as **optional**. These subjects complement the topic and give some additional information. However, completion of optional sections is left to the discretion of the teacher, who can take into account regional curriculum requirements.

G. BUNCH OF LINES (OPTIONAL)

Definition — bunch of lines

In the coordinate plane, a set of the lines pas called a **bunch of lines**.

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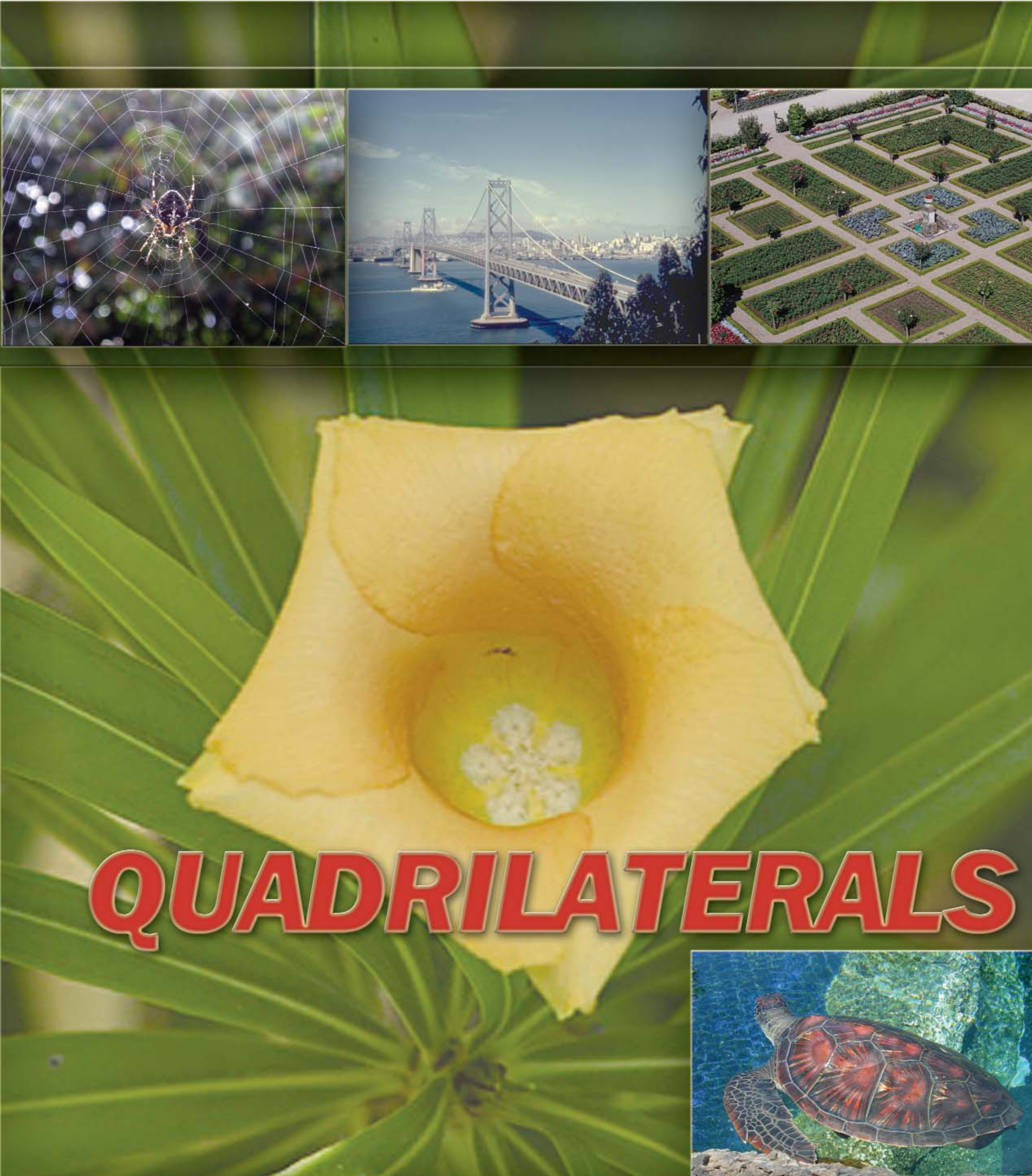


QUADRILATERALS

QUADRILATERALS

If you look around you, you will see many things which have four lines for sides. A book, a door, the spaces between the bars at a window, a slice of bread and the floor of a square room are all examples of a closed figure bounded by four line segments. A figure like this is called a **quadrilateral**. In other words, a quadrilateral is a geometrical figure which has four sides. In this section we will study quadrilaterals and their properties.

A. QUADRILATERALS AND THEIR BASIC PROPERTIES

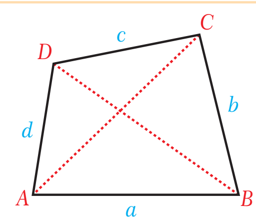
1. Definitions

Definition

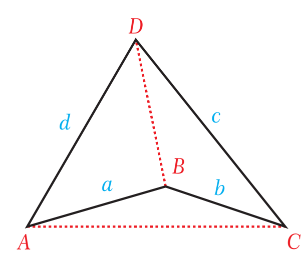
quadrilateral

A **quadrilateral** is a polygon which has four sides.

In each of the quadrilaterals $ABCD$ shown opposite, points A, B, C and D are the vertices and the line segments AB, BC, CD and DA are the sides of the quadrilateral. $\angle ABC$, $\angle BCD$, $\angle CDA$ and $\angle DAB$ are the interior angles of the quadrilateral. $\{A, C\}$ and $\{B, D\}$ are two examples of pairs of **opposite vertices**. The pairs of sides $\{AB, CD\}$ and $\{BC, DA\}$ are **opposite sides**. $\{\angle A, \angle C\}$ and $\{\angle B, \angle D\}$ are two pairs of **opposite angles**.



a convex quadrilateral



a concave quadrilateral

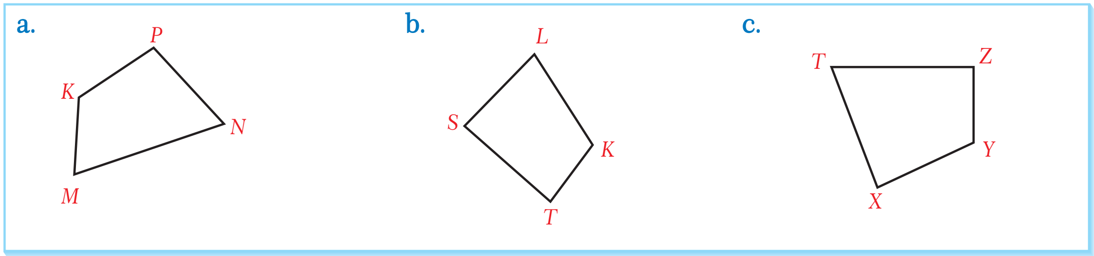
Since a quadrilateral is a polygon, it also has consecutive vertices, sides and angles.

In both figures, AC and BD are the diagonals of the quadrilateral. Notice that in a concave quadrilateral, one of the diagonals lies in its exterior region. In a convex quadrilateral, the diagonals always lie in its interior region.

The perimeter of a quadrilateral is the sum of the lengths of all of its sides. In other words, the perimeter of a quadrilateral $ABCD$ is $AB + BC + CD + DA$.

How many quadrilaterals can you see?

EXAMPLE 1 State the opposite sides, opposite angles and the diagonals in each quadrilateral.



Solution a. opposite sides: $\{KM, PN\}$ and $\{KP, MN\}$

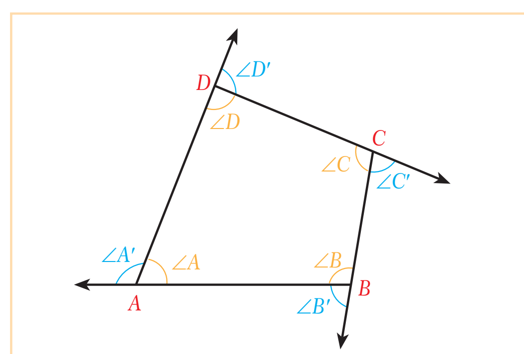
opposite angles: $\{\angle M, \angle P\}$ and $\{\angle K, \angle N\}$

diagonals: KN and MP

Questions b and c are left as an exercise for you.

2. Basic Properties of a Quadrilateral

a. Angles of a quadrilateral

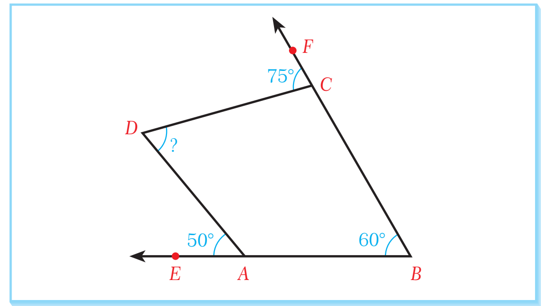


We have already seen that the sum of the measures of the interior angles of an n-sided polygon is $(n-2)\cdot 180°$. Since a quadrilateral has four sides we have $n = 4$, and $(4-2)\cdot 180° = 2\cdot 180° = 360°$. So the sum of the measures of the interior angles of a quadrilateral is **360°**.

The sum of the measures of the exterior angles of a quadrilateral is also 360° (since this is true for all polygons).

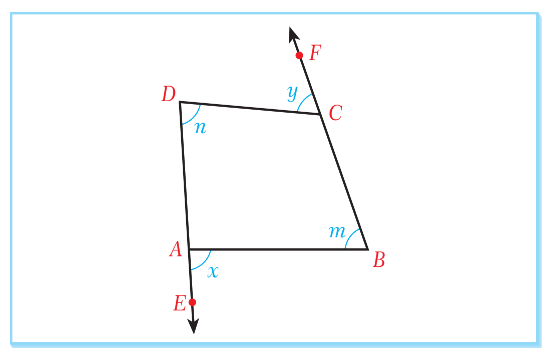
In conclusion, for any quadrilateral $ABCD$ we have $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360°$ and $m(\angle A') + m(\angle B') + m(\angle C') + m(\angle D') = 360°$.

EXAMPLE 2 In the figure,
$m(\angle EAD) = 50°$,
$m(\angle EBF) = 60°$ and
$m(\angle DCF) = 75°$.
Find the measure of $\angle ADC$.



Solution $m(\angle DAB) + m(\angle DAE) = 180°$ (supplementary angles)
$m(\angle DAB) = 180° - m(\angle DAE) = 180° - 50° = 130°$
$m(\angle BCD) + m(\angle DCF) = 180°$ (supplementary angles)
$m(\angle BCD) = 180° - m(\angle DCF) = 180° - 75° = 105°$
In quadrilateral $ABCD$,
$m(\angle ADC) + m(\angle DAB) + m(\angle ABC) + m(\angle BCD) = 360°$ (sum of the interior angles)
$m(\angle ADC) + 130° + 60° + 105° = 360°$
$m(\angle ADC) = 360° - 295°$
$= 65°$.

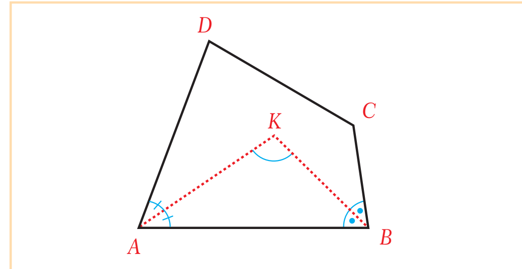
EXAMPLE 3 In the figure,
$m(\angle BAE) = x$,
$m(\angle ABF) = m$,
$m(\angle FCD) = y$,
$m(\angle CDE) = n$ and
$x + y = 105°$.
Calculate $m + n$.



Solution $m(\angle DAB) + x = 180°$; $m(\angle DAB) = 180° - x$ (supplementary angles)
$m(\angle BCD) + y = 180°$; $m(\angle BCD) = 180° - y$ (supplementary angles)
In quadrilateral $ABCD$,
$m(\angle CDA) + m(\angle DAB) + m(\angle ABC) + m(\angle BCD) = 360°$ (sum of interior angles)
$n + (180° - x) + m + (180° - y) = 360°$ (substitution)
$m + n = 360° - 360° + x + y$; $m + n = x + y = 105°$. ($x + y = 105°$ is given)

Property

In a quadrilateral, the measure of the angle formed by the bisectors of two consecutive interior angles equals the half the sum of the measures of the other two angles.

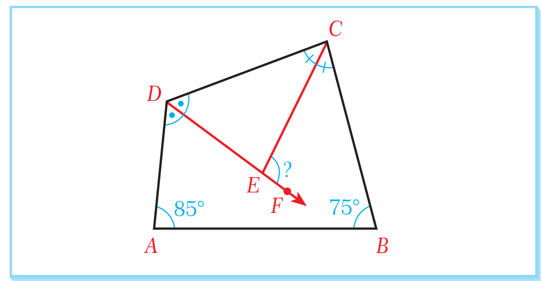


In the figure, $ABCD$ is a quadrilateral and AK and BK are the bisectors of $\angle A$ and $\angle B$ respectively. So by Property 2,

$$m(\angle AKB) = \frac{m(\angle C) + m(\angle D)}{2}.$$

EXAMPLE 4

In the figure, DF and CE are bisectors of $\angle D$ and $\angle C$ respectively. Given that $m(\angle A) = 85°$ and $m(\angle B) = 75°$, find the measure of $\angle CEF$.



Solution 1 $\angle D$ and $\angle C$ are consecutive interior angles. By Property 2,

$$m(\angle DEC) = \frac{m(\angle A) + m(\angle B)}{2} = \frac{85° + 75°}{2} = 80°.$$

$$m(\angle CEF) + m(\angle DEC) = 180° \qquad \text{(supplementary angles)}$$

$$\begin{aligned} m(\angle CEF) &= 180° - m(\angle DEC) \\ &= 180° - 80° \\ &= 100° \end{aligned}$$

Solution 2 In quadrilateral $ABCD$,

$$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360° \qquad \text{(sum of interior angles)}$$

$$m(\angle C) + m(\angle D) = 360° - 85° - 75° = 200°.$$

In ΔCDE,

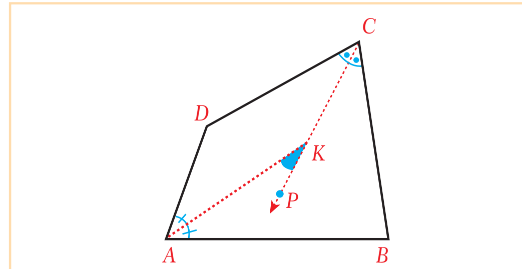
$$m(\angle CDE) = \frac{m(\angle CDA)}{2} \text{ and } m(\angle DCE) = \frac{m(\angle DCB)}{2} \qquad (DE \text{ and } CE \text{ are bisectors})$$

$$\begin{aligned} m\angle(CEF) &= m\angle(CDE) + m(\angle DCE) \qquad \text{(exterior angle property of a triangle)} \\ &= \frac{m(\angle CDA)}{2} + \frac{m(\angle DCB)}{2} \\ &= \frac{m(\angle DCB) + m(\angle CDA)}{2} = \frac{200°}{2} = 100°. \end{aligned}$$

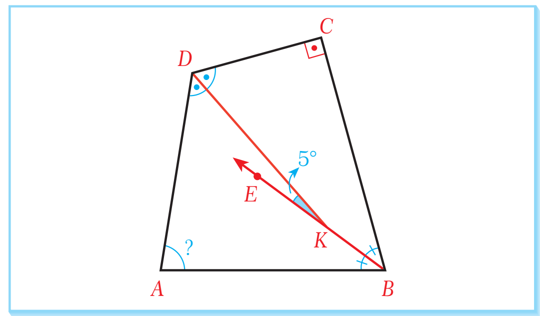
Property

In a quadrilateral, the measure of the acute angle formed by the bisectors of opposite angles is half the absolute value of the difference between the measures of the other two angles. For example, in the figure,

$$m(\angle AKP) = \frac{|m(\angle D) - m(\angle B)|}{2}.$$



EXAMPLE 5 In the figure, DK and BE are bisectors of $\angle D$, $\angle B$, respectively. Given that $m(\angle C) > m(\angle A)$, $m(\angle C) = 90°$ and $m(\angle DKE) = 5°$, find $m(\angle A)$.



Solution 1 By Property 3,

$$m(\angle DKE) = \frac{|m(\angle C) - m(\angle A)|}{2}$$

$$5° = \frac{90° - m(\angle A)}{2}. \qquad (m(\angle C) > m(\angle A))$$

So $m(\angle A) = 80°$.

Solution 2 $m(\angle DKE) + m(\angle DKB) = 180°$ (supplementary angles)

$$m(\angle DKB) = 180° - m(\angle DKE)$$

$$= 180° - 5° = 175°$$

In quadrilateral $BCDK$,

$m(\angle C) + m(\angle CDK) + m(\angle CBK) + m(\angle BKD) = 360°$ (sum of interior angles)

$$m(\angle CDK) + m(\angle CBK) = 360° - 90° - 175° = 95°$$

$$= \frac{m(\angle D)}{2} + \frac{m(\angle B)}{2}$$

$$= \frac{m(\angle D) + m(\angle B)}{2}$$

$$\frac{m(\angle D) + m(\angle B)}{2} = 95°$$

$$m(\angle D) + m(\angle B) = 190°.$$

In quadrilateral $ABCD$,

$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360°$ (sum of interior angles)

$$m(\angle A) = 360° - 90° - 190° = 80°.$$

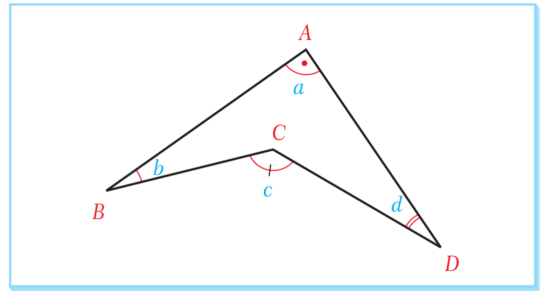
EXAMPLE 6 In the figure, $ABCD$ is a concave quadrilateral with $m(\angle BAD) = a$,

$m(\angle ABC) = b$,

$m(\angle BCD) = c$ and

$m(\angle CDA) = d$.

Show that $c = a + b + d$.



Solution Let us extend the line segment BC so that it intersects side AD at point K.

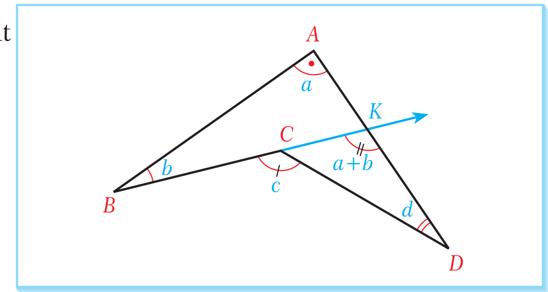
$\angle BKD$ is an exterior angle of ΔABK.

So $m(\angle BKD) = a + b$.

Also, $\angle BCD$ is an exterior angle of ΔCKD.

So $m(\angle BCD) = m(\angle BKD) + d = a + b + d$.

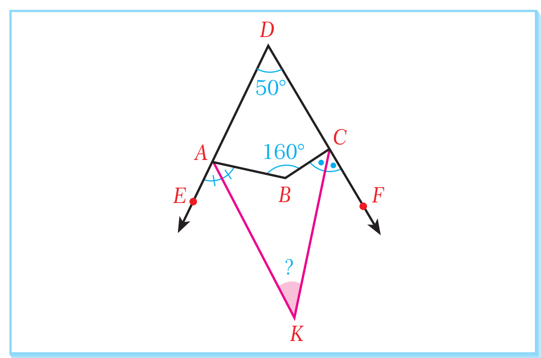
So $c = a + b + d$.



EXAMPLE 7 In the figure, AK and CK bisect $\angle EAB$ and $\angle BCF$ respectively.

If $m(\angle B) = 160°$ and $m(\angle D) = 50°$,

find $m(\angle AKC)$.



Solution In quadrilateral $ABCD$,

$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360°$ (sum of interior angles)

$m(\angle A) + m(\angle C) = 360° - 50° - 160° = 150°$

$m(\angle BAE) + m(\angle BAD) = 180°$ (supplementary angles)

$m(\angle BAE) = 180° - m(\angle BAD)$

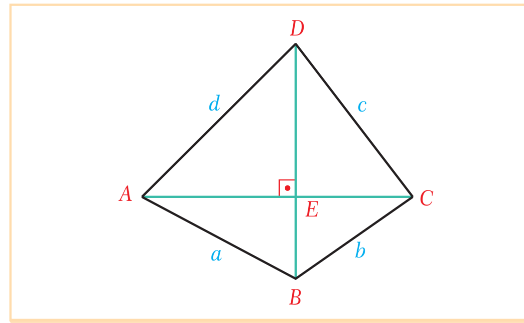
$m(\angle BCF) + m(\angle BCD) = 180°$ (supplementary angles)

$m(\angle BCF) = 180° - m(\angle BCD)$.

b. Sides of a quadrilateral

Property

If the diagonals of a quadrilateral are perpendicular to each other, the sums of the squares of the lengths of opposite sides of the quadrilateral are equal.



In the figure, $ABCD$ is a quadrilateral and the diagonals AC and BD are perpendicular to each other. So by Property 4,

$$AB^2 + DC^2 = AD^2 + BC^2.$$

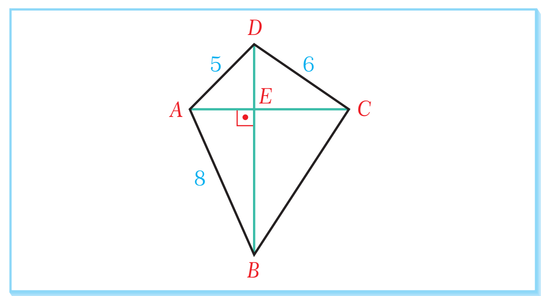
EXAMPLE 8 The figure shows a quadrilateral $ABCD$ whose diagonals are perpendicular.

$AB = 8$ cm,

$AD = 5$ cm and

$DC = 6$ cm are given.

Find the length of side BC.



Solution By Property 4 we can write $AB^2 + DC^2 = AD^2 + BC^2$.

Substituting the given values into this equation gives

$$8^2 + 6^2 = 5^2 + BC^2$$

$$BC^2 = 64 + 36 - 25$$

$$BC^2 = 75$$

$$BC = 5\sqrt{3} \text{ cm.}$$

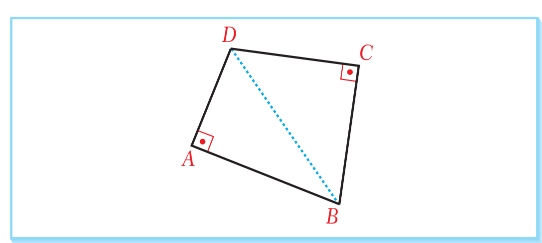
EXAMPLE 9 In a quadrilateral $ABCD$, $m(\angle A) = m(\angle C) = 90°$. Show that $AB^2 + AD^2 = BC^2 + CD^2$.

Solution Look at the figure. Drawing the diagonal BD creates two right triangles ΔDAB and ΔDCB.

By the Pythagorean Theorem,

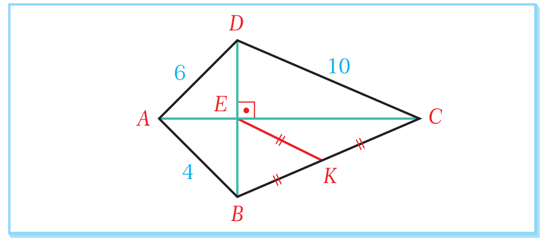
$AB^2 + AD^2 = BD^2$ and $BC^2 + CD^2 = BD^2$.

So $AB^2 + AD^2 = BC^2 + CD^2$, as required.



EXAMPLE 10 The diagonals of a quadrilateral $ABCD$ are perpendicular, with $AB = 4$ cm, $AD = 6$ cm and $DC = 10$ cm. Point E is the intersection point of the diagonals, and point K is on side BC such that that $BK = KC$. What is the length of EK?

Solution We begin by drawing the figure.



Since the diagonals are perpendicular,

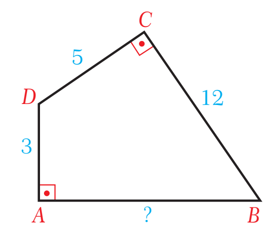
$$AB^2 + DC^2 = AD^2 + BC^2 \text{ (by Property 4)}$$
$$4^2 + 10^2 = 6^2 + BC^2$$
$$BC^2 = 116 - 36$$
$$BC^2 = \sqrt{80}$$
$$BC = 4\sqrt{5} \text{ cm.}$$

In a right triangle, the length of the median drawn to the hypotenuse is half the length of hypotenuse.

EK is a median of the right triangle BEC, and since the length of the median drawn to the hypotenuse is half the length of the hypotenuse, $EK = \frac{BC}{2} = \frac{4\sqrt{5}}{2} = 2\sqrt{5}$ cm.

Check Yourself

1. In the figure, $m(\angle C) = m(\angle A) = 90°$. Given $AD = 3$ cm, $DC = 5$ cm and $CB = 12$ cm, find the length of AB.



2. The diagonals of a quadrilateral are perpendicular to each other. The lengths of two opposite sides are 8 cm and 4 cm, and the ratio of the lengths of the other two opposite sides is 1:2. Find the lengths of the unknown sides.

Answers

1. $4\sqrt{10}$ cm 2. 4 cm and 8 cm

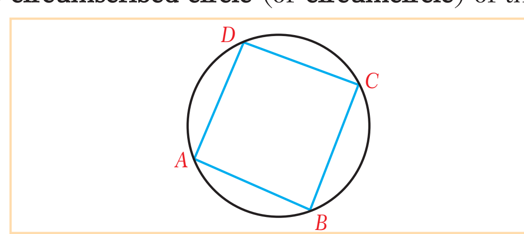
3. Inscribed and Circumscribed Quadrilaterals

Definition

inscribed quadrilateral, cyclic quadrilateral

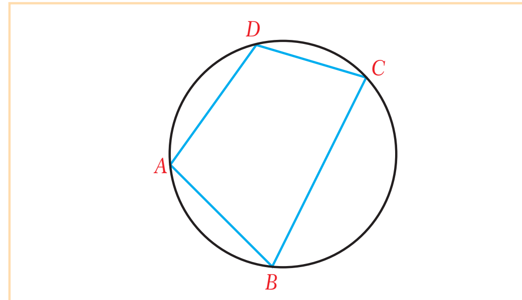
A quadrilateral is called an **inscribed quadrilateral** (or **cyclic quadrilateral**) if all of its vertices lie on the same circle. This circle is called the **circumscribed circle** (or **circumcircle**) of the quadrilateral.

In the figure, $ABCD$ is an inscribed quadrilateral.



Property

The sum of the measures of either pair of opposite angles of an inscribed quadrilateral is 180°.



In the figure, $ABCD$ is an inscribed quadrilateral. So by Property 5,

$$m(\angle A) + m(\angle C) = 180° \text{ and}$$

$$m(\angle B) + m(\angle D) = 180°.$$

In fact, if the sum of any pair of opposite angles in a quadrilateral is equal to 180° then the quadrilateral is always an inscribed quadrilateral.

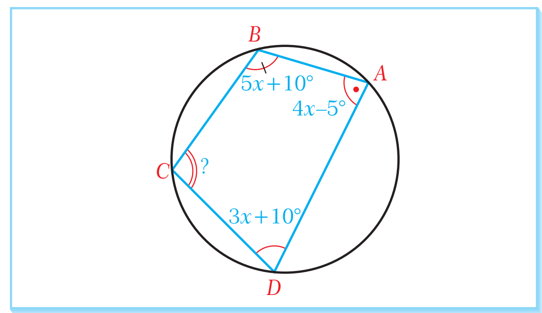
EXAMPLE 11 In the figure, $ABCD$ is an inscribed quadrilateral with

$m(\angle A) = 4x - 5°$,

$m(\angle B) = 5x + 10°$ and

$m(\angle D) = 3x + 10°$.

Find $m(\angle C)$.



Solution Since $ABCD$ is an inscribed quadrilateral, by Property 5 its opposite angles are supplementary. So

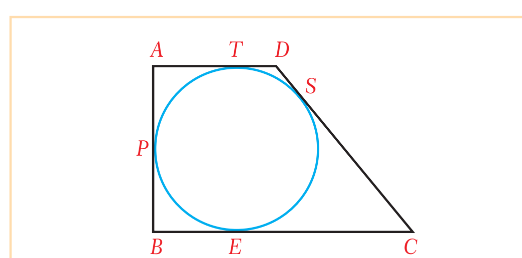
$$\begin{aligned} m(\angle B) + m(\angle D) &= 180° \\ 5x + 10° + 3x + 10° &= 180° \\ 8x &= 160° \\ x &= 20°, \text{ and} \\ m(\angle A) + m(\angle C) &= 180° \\ (4 \cdot 20°) - 5° + m(\angle C) &= 180° \\ 75° + m(\angle C) &= 180° \\ m(\angle C) &= 105°. \end{aligned}$$

Definition

circumscribed quadrilateral

A quadrilateral is called a **circumscribed quadrilateral** if all of its sides are tangent to the same circle. This circle is called the **inscribed circle** of the quadrilateral.

In the figure, $ABCD$ is a circumscribed quadrilateral.



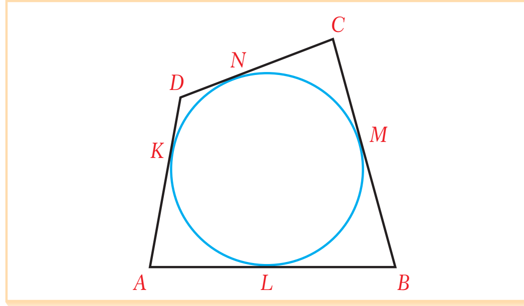
Property

The sums of the lengths of the opposite sides of a circumscribed quadrilateral are equal.

In the figure, $ABCD$ is a circumscribed quadrilateral. So by Property 6,

$$AB + CD = BC + AD.$$

In fact, if the sums of the lengths of the opposite sides of a quadrilateral are equal then the quadrilateral is always a circumscribed quadrilateral.



EXAMPLE 12 Three consecutive sides of a circumscribed quadrilateral measure 9 cm, 12 cm and 13 cm respectively. Find the length of the fourth side.

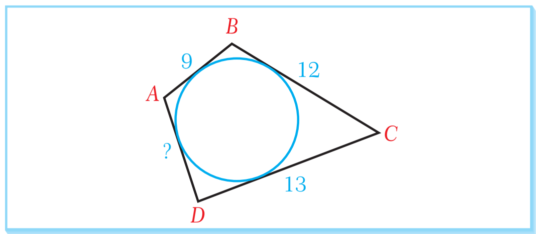
Solution Look at the figure. Let $AB = 9$ cm, $BC = 12$ cm and $CD = 13$ cm.

Since $ABCD$ is a circumscribed quadrilateral, by Property 6 we have

$$AB + CD = BC + AD$$

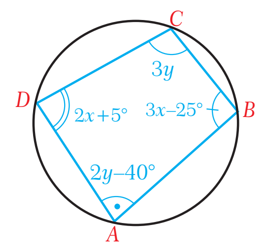
$$9 + 13 = 12 + AD$$

$$AD = 10 \text{ cm}.$$



Check Yourself

1. In the figure, $m(\angle A) = 2y - 40°$, $m(\angle B) = 3x - 25°$, $m(\angle D) = 2x + 5°$ and $m(\angle C) = 3y$.
   Find the values of x and y.



2. Three consecutive sides of a circumscribed quadrilateral measure 6 cm, 8 cm and 9 cm respectively. Find the length of the fourth side.

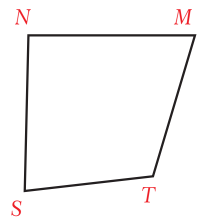
Answers

1. $x = 40°$, $y = 44°$ 2. 7 cm

EXERCISES 1.1

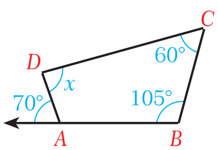
A. Quadrilaterals and Their Basic Properties

1. State the pairs of opposite and consecutive sides and angles and the diagonals in the polygon opposite.

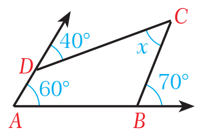


2. Find the measure of angle x in each figure.

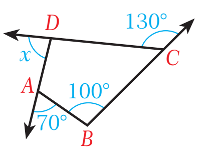
a.



b.

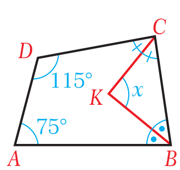


c.

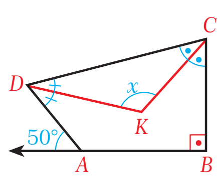


3. Find the measure of angle x in each figure.

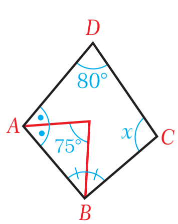
a.



b.

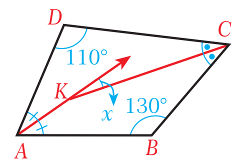


c.

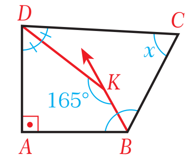


4. Each figure shows the bisectors of opposite angles of a quatrilateral. Find the measure of angle x in each case.

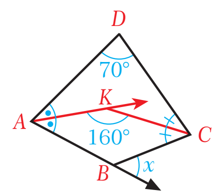
a.



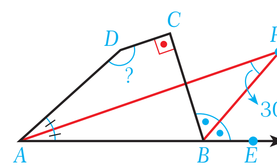
b.



c.

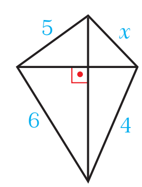


5. In the figure, $m(\angle C) = 90°$, AP and BP are respectively bisectors of $\angle DAB$ and $\angle CBE$, and $m(\angle P) = 30°$. Find $m(\angle ADC)$.

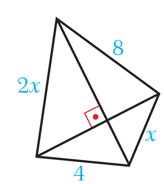


6. Find the length x in each figure.

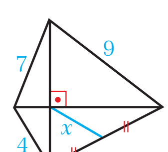
a.



b.

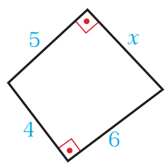


c.

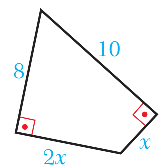


7. Find the value of x in each figure.

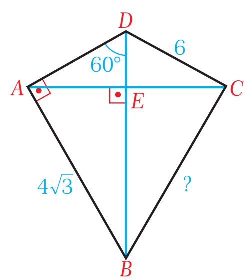
a.



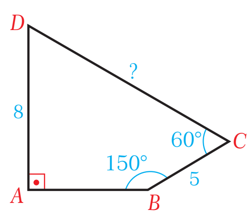
b.



8. In the figure, $AC \perp BD$ and $AD \perp AB$, $m(\angle ADB) = 60°$, $AB = 4\sqrt{3}$ cm and $DC = 6$ cm. Find the length of BC.



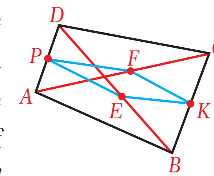
9. ✪ In the figure, $m(\angle A) = 90°$, $m(\angle B) = 150°$, $m(\angle C) = 60°$, $AD = 8$ cm and $BC = 5$ cm. Find the length of DC.



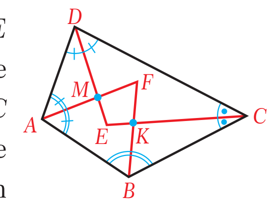
10. $ABCD$ is an inscribed quadrilateral with $m(\angle A) = 70°$ and $m(\angle B) = 100°$. Find $m(\angle C)$ and $m(\angle D)$.

11. A quadrilateral $ABCD$ is a circumscribed quadrilateral with $AB = 9$ cm, $BC = 7$ cm and $CD = 10$ cm. Find the length of side AD.

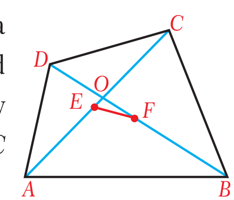
12. ✪ In the figure, AC and BD are diagonals of the quadrilateral $ABCD$. Points P and K are respectively the midpoints of sides AD and BC, and points F and E are respectively the midpoints of diagonals AC and BD. Show that $P(EKFP) = AB + DC$.



13. ✪ In the figure, AF, BF, CE and DE are respectively the bisectors of $\angle A$, $\angle B$, $\angle C$ and $\angle D$. Prove that the quadrilateral $EKFM$ is an inscribed quadrilateral.



14. ✪✪ In the figure, $ABCD$ is a convex quadrilateral and points E and F are respectively the midpoints of diagonals AC and BD.



Prove that

$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + (4 \cdot EF^2)$.

(Hint: Use the property of the length of a median in ΔBDA, ΔBCD and $\angle AFC$.)

15. In the quadrilateral $ABCD$ opposite,

$AE = EC = 10$ cm,

$BF = FD = \dfrac{10\sqrt{7}}{2}$ cm,

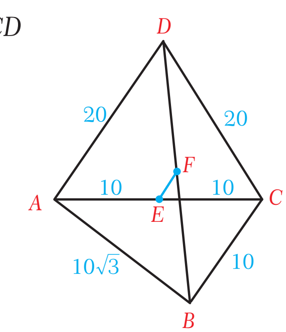
$BC = 10$ cm,

$DC = AD = 20$ cm and

$AB = 10\sqrt{3}$ cm.

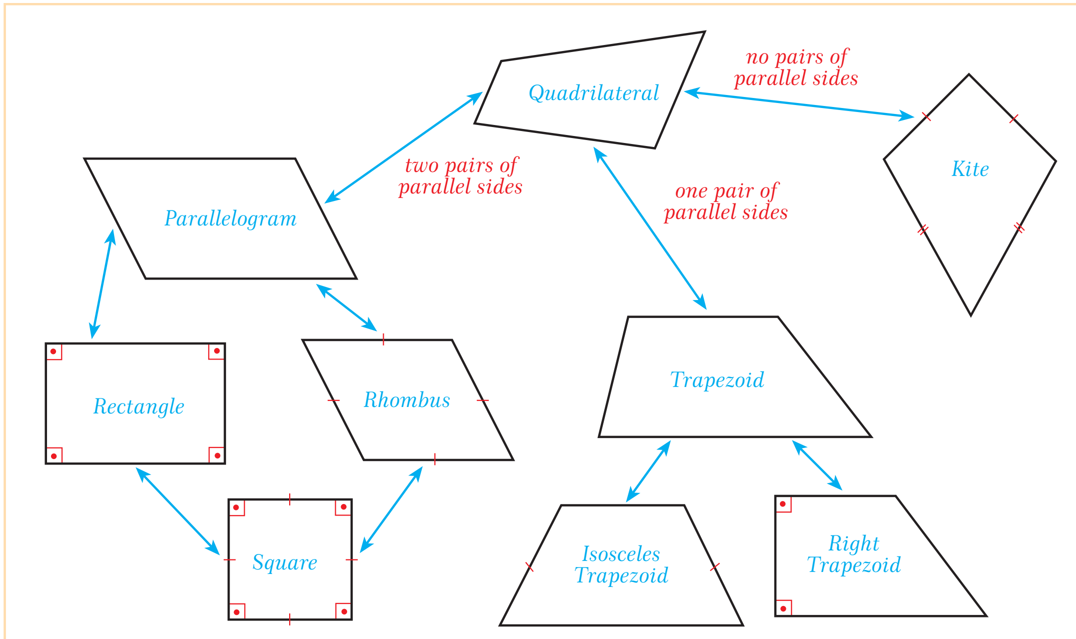
Find the length of EF.

(Hint: Use the formula given in question 14.)



There are many different types of quadrilateral, but they all have several things in common: all of them have four sides and two diagonals, and the sum of the measures of their interior angles is 360°. This how they are alike, but what makes them different?

The figure shows some special types of quadrilateral and the relationships between them. In this section we will look at the properties of each of these special quadrilaterals in turn.



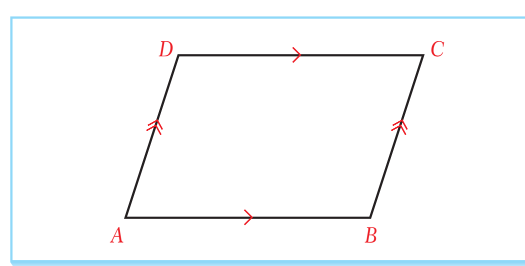
B. PARALLELOGRAM

1. Definition

Definition

parallelogram

A **parallelogram** is a quadrilateral which has two pairs of opposite parallel sides.



In the figure, $AB \parallel DC$ and $BC \parallel AD$.

So quadrilateral $ABCD$ is a parallelogram by definition.

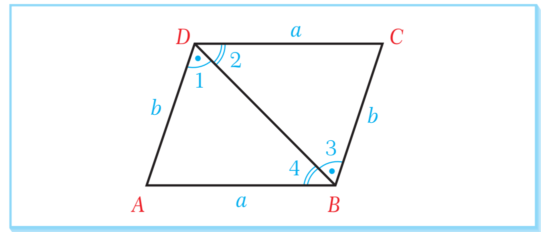
2. Properties of a Parallelogram

Theorem 5

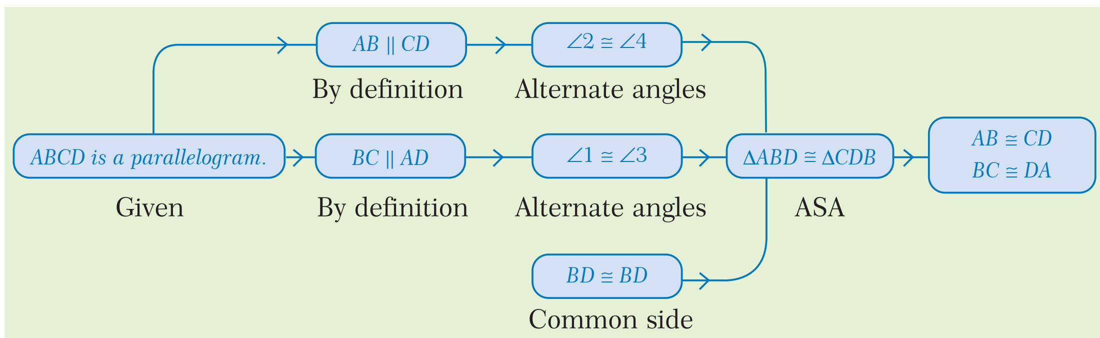
Opposite sides of a parallelogram are congruent.

Proof Look at the figure. Given that $ABCD$ is a parallelogram, we need to show $AB \cong CD$ and $BC \cong DA$.

Let us use a flow chart proof.



> **ASA** means the **Angle Side Angle postulate:** If two angles and their common side in a triangle are congruent to two angles and their common side in another triangle, then the triangles are congruent.



So opposite sides of a parallelogram are congruent, as required.

Notice that as a result of Theorem 5, the perimeter of a parallelogram is twice the sum of any two consecutive sides:

$$P(ABCD) = 2 \cdot (AB + BC).$$

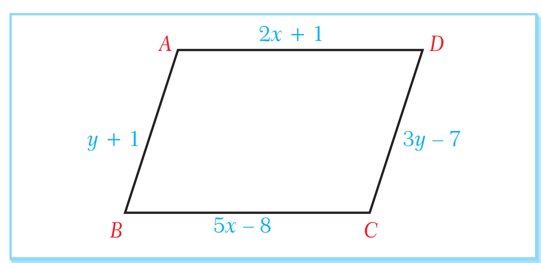
EXAMPLE 13 In the figure, $ABCD$ is a parallelogram with

$AB = y + 1$,

$BC = 5x - 8$,

$CD = 3y - 7$ and $AD = 2x + 1$.

Find the lengths of the sides of the parallelogram.



Solution Since the lengths of opposite sides of a parallelogram are equal, $AB = CD$ and $BC = AD$.

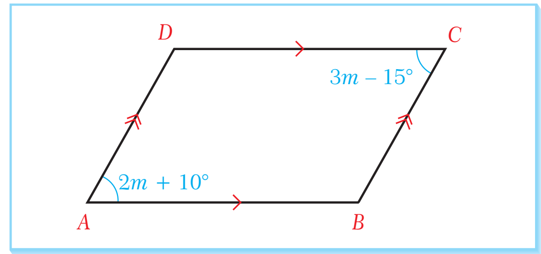
So $y + 1 = 3y - 7$, and $2x + 1 = 5x - 8$

$2y = 8$ $\quad$ $3x = 9$

$y = 4$ cm $\quad$ $x = 3$ cm.

So $AB = CD = 5$ cm and $BC = AD = 7$ cm.

EXAMPLE 14 In the figure, $ABCD$ is a parallelogram with $m(\angle A) = 2m + 10°$ and $m(\angle C) = 3m - 15°$. Find the measures of the interior angles of the parallelogram.



Solution The measures of opposite angles in a parallelogram are equal:

$$m(\angle A) = m(\angle C)$$
$$2m + 10° = 3m - 15°$$
$$m = 25°.$$

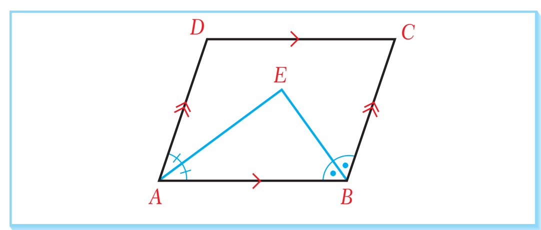
So $m(\angle A) = m(\angle C) = 60°$.

Since consecutive angles in a parallelogram are supplementary, we have

$$m(\angle A) + m(\angle D) = 180°$$
$$60° + m(\angle D) = 180°$$
$$m(\angle D) = 120°$$

$m(\angle B) = m(\angle D) = 120°$. (opposite angles)

EXAMPLE 15 Show that the measure of the angle formed by the bisectors of any two consecutive angles in a parallelogram is 90°.



Solution In the figure, point E is the intersection point of the bisectors of $\angle A$ and $\angle B$. We need to show $m(\angle E) = 90°$. We know

$m(\angle A) + m(\angle B) = 180°$, (supplementary angles)

$m(\angle EAB) = \dfrac{m(\angle A)}{2}$ and,

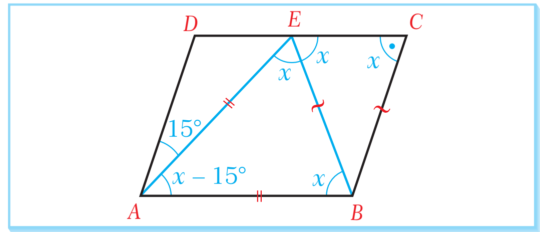
$m(\angle EBA) = \dfrac{m(\angle B)}{2}$. Adding these last two equations side by side gives us

$$m(\angle EAB) + m(\angle EBA) = \frac{m(\angle A)}{2} + \frac{m(\angle B)}{2} = \frac{m(\angle A) + m(\angle B)}{2} = \frac{180°}{2} = 90°.$$

In ΔAEB,

$$m(\angle EAB) + m(\angle EBA) + m(\angle E) = 180° \qquad \text{(sum of interior angles)}$$
$$90° + m(\angle E) = 180°$$
$$m(\angle E) = 90°.$$

EXAMPLE 16 $ABCD$ is a parallelogram with $AB > AD$. Point E is on the side DC such that $BE = BC$, $AE = AB$ and $m(\angle DAE) = 15°$. Find the measures of the interior angles of the parallelogram.



Solution We begin by drawing the figure. Let $m(\angle C) = x$. Then

$m(\angle A) = m(\angle C) = x$ (opposite angles)

$m(\angle EAB) = m(\angle C) - m(\angle DAE) = x - 15°$

$m(\angle BCE) = m(\angle CEB) = x$ (base angles in isosceles triangle BEC)

$m(\angle ABE) = m(\angle CEB) = x$ (alternate interior angles, $DC \parallel AB$)

$m(\angle BEA) = m(\angle ABE) = x.$ (base angles in isosceles triangle ABE)

In ΔABE,

$$m(\angle EAB) + m(\angle ABE) + m(\angle BEA) = 180° \quad \text{(sum of interior angles)}$$
$$x - 15° + x + x = 180°$$
$$3x = 195°$$
$$x = 65°.$$

So $m(\angle A) = m(\angle C) = 65°$.

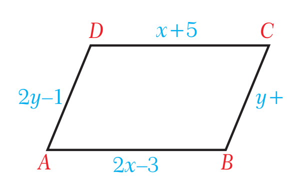
Since consecutive angles in a parallelogram are supplementary, we have

$$m(\angle A) + m(\angle B) = 180°$$
$$65° + m(\angle B) = 180°$$
$$m(\angle B) = 115°.$$

Since $\angle B$ and $\angle D$ are opposite angles, $m(\angle D) = m(\angle B) = 115°$.

Check Yourself

1. In the figure, $ABCD$ is a parallelogram with $AD = 2y - 1$, $AB = 2x - 3$, $BC = y + 1$ and $CD = x + 5$. Find the perimeter of the parallelogram.



2. The measure of the angle between one side of a parallelogram and the altitude drawn from one of its obtuse angles is 35°. Find the measures of the interior angles of the parallelogram.

Answers

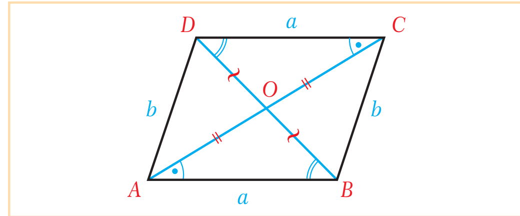
1. 32 2. 55° and 125°

Theorem 8

The diagonals of a parallelogram bisect each other.

Proof

Look at the figure.



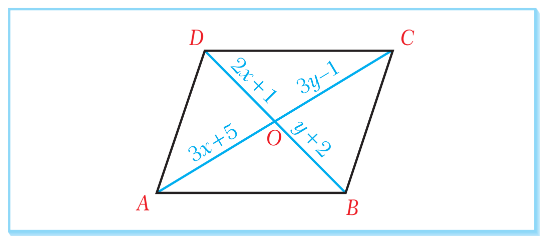
Given that $ABCD$ is a parallelogram, we need to show $AO \cong OC$ and $BO \cong OD$.

Let us prove it with a two-column proof.

Statements	Reasons
1. $ABCD$ is a parallelogram.	Given
2. $AB \parallel DC$	Definition of a parallelogram
3. $\angle OBA \cong \angle ODC$ and $\angle OAB \cong \angle OCD$	Alternate interior angles
4. $AB \cong DC$	Opposite sides are congruent.
5. $\Delta OBA \cong \Delta ODC$	ASA by 3 and 4
6. $AO \cong OC$ and $BO \cong OD$	Corresponding sides of congruent triangles

EXAMPLE 17

In the figure, $ABCD$ is a parallelogram and point O is the intersection of diagonals AC and DB. $AO = 3x + 5$, $OC = 3y - 1$, $BO = y + 2$ and $OD = 2x + 1$ are given.



Find the lengths of the diagonals AC and BD.

Solution

The diagonals of a parallelogram bisect each other, so $AO = OC$ and $BO = OD$.

This gives the system

$$\begin{cases} 3x+5 = 3y-1 \\ y+2 = 2x+1 \end{cases} ; \quad \begin{cases} 3x-3y = -6 \\ y = 2x-1. \end{cases}$$

Substitute $y = 2x - 1$ in the first equation:

$$\begin{aligned} 3x - 3(2x - 1) &= -6 \\ 3x - 6x + 3 &= -6 \\ -3x &= -9 \\ x &= 3. \end{aligned}$$

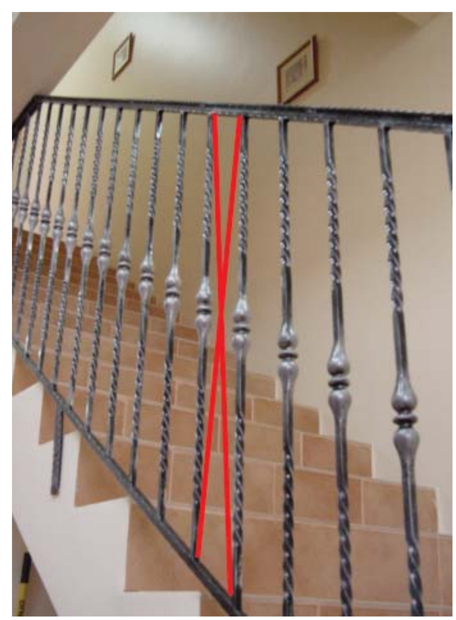
For $x = 3$, $y = 2x - 1$

$$\begin{aligned} y &= 2 \cdot 3 - 1 \\ y &= 5. \end{aligned}$$

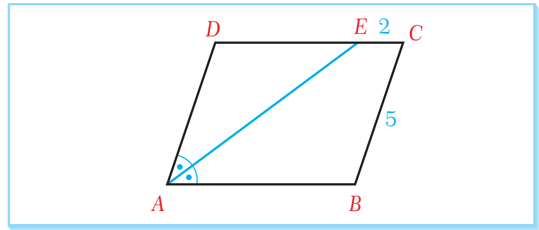
So $x = 3$ and $y = 5$, and

$AC = 2 \cdot AO = 2(3x + 5) = 2(3 \cdot 3 + 5) = 28$,

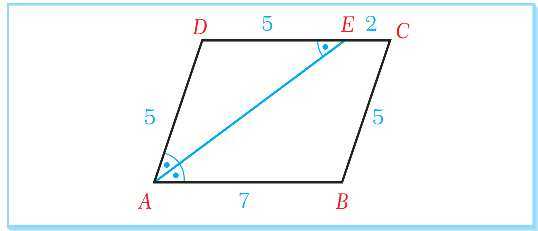
$BD = 2 \cdot BO = 2(y + 2) = 2(5 + 2) = 14$.



EXAMPLE 18 In the figure, $ABCD$ is a parallelogram and AE is the bisector of $\angle A$. Given that $BC = 5$ cm and $EC = 2$ cm, find the perimeter of $ABCD$.



Solution The lengths of opposite sides of a parallelogram are equal, so $BC = AD = 5$ cm.



Opposite sides of a parallelogram are parallel, so $DC \parallel AB$. Now we can write

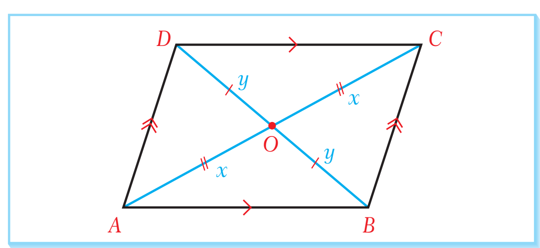
$m(\angle EAB) = m(\angle AED)$	(alternate interior angles)
ΔADE is isosceles	(two congruent angles in ΔADE)
$AD = DE = 5$ cm	(legs of isosceles ΔADE)
$DC = DE + EC$	
$DC = 5 + 2$	
$DC = 7$ cm	
$AB = DC = 7$ cm.	(opposite sides)
$P(ABCD) = 2 \cdot (7 + 5) = 24$ cm.	

EXAMPLE 19 In a parallelogram $ABCD$, $DC = 12$ cm and point O is the intersection point of the diagonals AC and DB. The perimeter of ΔCOB is 24 cm and the perimeter of ΔAOB is 28 cm. Find the perimeter of $ABCD$.

Solution $AB = DC = 12$ cm because opposite sides of a parallelogram are congruent.

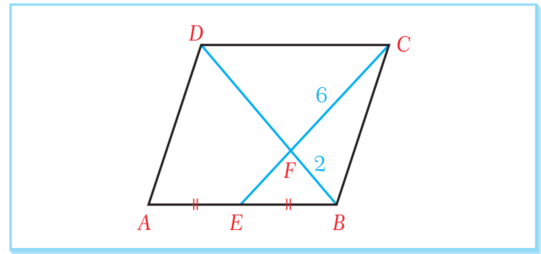
The diagonals of a parallelogram bisect each other, so let $AO = OC = x$ and $DO = OB = y$. Then

$$P(\Delta AOB) = AO + OB + AB$$
$$28 = x + y + 12$$
$$x + y = 16 \text{ cm, and}$$
$$P(\Delta COB) = CO + OB + BC$$
$$24 = x + y + BC$$
$$24 = 16 + BC$$
$$BC = 8 \text{ cm.}$$



So the perimeter of $ABCD$ is $2(AB + BC) = 2(12 + 8) = 40$ cm.

EXAMPLE 20 In the figure, $ABCD$ is a parallelogram. Point E is the midpoint of side AB and point F is the intersection of line segments EC and DB. Given $FB = 2$ cm and $FC = 6$ cm, find the lengths of DF and EF.

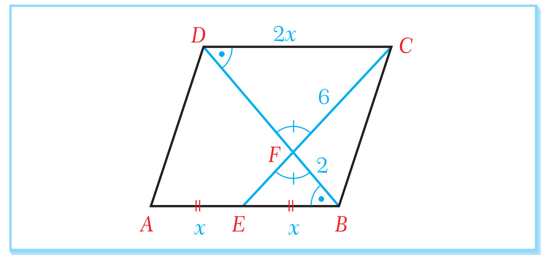


Solution $AE = EB$ since point E is the midpoint of AB.

Let us write $AE = EB = x$, so $AB = 2x$.

The lengths of opposite sides of a parallelogram are equal, so $CD = AB = 2x$.

Opposite sides of a parallelogram are parallel, so $DC \parallel AB$. Also,



$m(\angle DFC) = m(\angle BFE)$ (vertical angles)

$m(\angle FBE) = m(\angle FDC)$. (alternate interior angles)

Angle Angle (AA) similarity postulate: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

So $\Delta FEB \sim \Delta FCD$ by the Angle Angle similarity postulate.

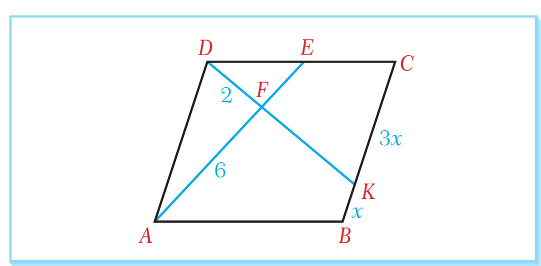
If the triangles are similar, then the lengths of their corresponding sides are proportional, so

$\frac{FE}{FC} = \frac{EB}{CD} = \frac{FB}{FD}$, $\frac{FE}{6} = \frac{x}{2x} = \frac{2}{FD}$.

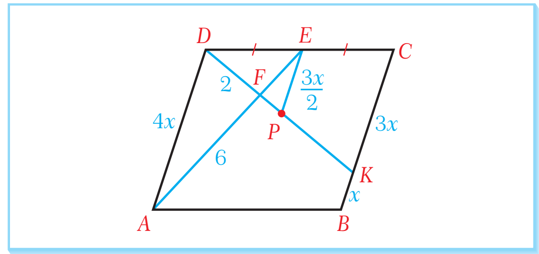
Since $\frac{FE}{6} = \frac{x}{2x}$, by simplification and cross multiplication we get $EF = FE = 3$ cm.

Similarly, $\frac{x}{2x} = \frac{2}{FD}$ gives us $DF = FD = 4$ cm.

EXAMPLE 21 In the figure, $ABCD$ is a parallelogram. Point E is the midpoint of side DC and point K is on side BC such that $KC = 3 \cdot KB$. Point F is the intersection of line segments AE and DK. $DF = 2$ cm and $AF = 6$ cm are given. Find the lengths of FK and EF.



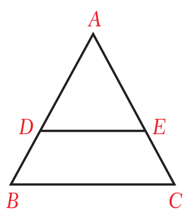
Solution Let $KB = x$. So $KC = 3x$.



The lengths of opposite sides of a parallelogram are equal, so $CB = AD = 4x$.

Let P be a point on the line segment DK and let us draw the line segment EP such that $EP \parallel AD$.

Triangle proportionality theorem: A line parallel to one side of a triangle which intersects the other two sides divides the two sides proportionally.



$DE \parallel BC \Rightarrow \dfrac{AD}{DB} = \dfrac{AE}{EC}$

So $EP \parallel BC$, because $AD \parallel BC$.

Since point E is the midpoint of DC and $EP \parallel BC$, by the triangle proportionality theorem we can say that point P is the midpoint of DK and so EP is a midsegment of ΔCDK.

So $EP = \dfrac{KC}{2} = \dfrac{3x}{2}$. Also,

$m(\angle FAD) = m(\angle FEP)$ (alternate interior angles)

$m(\angle EFP) = m(\angle AFD)$. (vertical angles)

So $\Delta FEP \sim \Delta FAD$ by the Angle Angle similarity postulate.

If the triangles are similar then the lengths of their corresponding sides are proportional, i.e.

$$\frac{FE}{FA} = \frac{EP}{AD} = \frac{FP}{FD};\ \frac{FE}{6} = \frac{\frac{3x}{2}}{4x} = \frac{FP}{2}.$$

So $\dfrac{FE}{6} = \dfrac{\frac{3x}{2}}{4x}$, which gives us $FE = \dfrac{9}{4}$ cm.

Similarly, $\dfrac{\frac{3x}{2}}{4x} = \dfrac{FP}{2}$ which gives us $FP = \dfrac{3}{4}$ cm. Now

$$DP = DF + FP$$

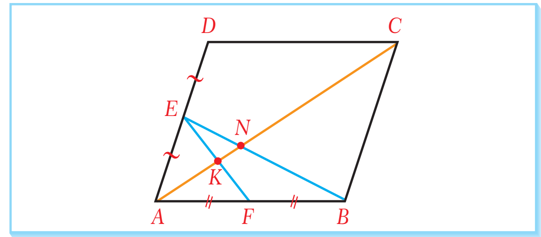
$$DP = 2 + \frac{4}{3} = 3\frac{1}{3} \text{ cm}$$

$PK = DP = 3\dfrac{1}{3}$ cm, and so finally (point P is the midpoint of DK)

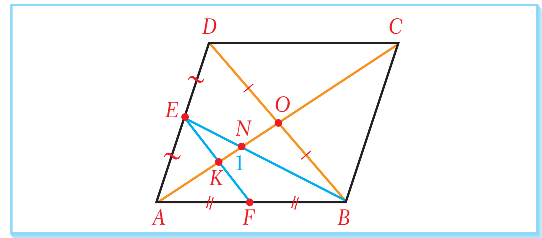
$$FK = PK + FP$$

$$= 3\frac{1}{3} + \frac{4}{3} = 4\frac{2}{3} \text{ cm}.$$

EXAMPLE 22 In the figure, $ABCD$ is a parallelogram. Points E and F are the midpoints of sides AD and AB respectively, and point K is the intersection of EF and AC. Point N is the intersection of EB and AC. If $KN = 1$ cm, find the length of AC.



Solution Let point O be the intersection of the diagonals AC and DB. Since the diagonals of a parallelogram bisect each other, $AO = OC$ and $DO = OB$. Also, $EF \parallel BD$ since EF is a midsegment of ΔADB.



So EK is the midsegment of ΔADO, which means $EK = \dfrac{OD}{2}$, i.e. $EK = \dfrac{BO}{2}$ (since $OD = BO$).

Now

$$m(\angle NEK) = m(\angle NBO) \quad \text{(alternate interior angles)}$$

$$m(\angle ENK) = m(\angle BNO). \quad \text{(vertical angles)}$$

So $\Delta ENK \sim \Delta BNO$ by the Angle Angle similarity postulate.

If two triangles are similar then their corresponding sides are proportional:

$$\frac{EN}{BN} = \frac{EK}{BO} = \frac{KN}{NO}, \ \frac{NK}{BN} = \frac{\frac{OB}{2}}{BO} = \frac{1}{NO}.$$

So $\dfrac{\frac{OB}{2}}{BO} = \dfrac{1}{NO}$, which gives us $NO = 2$ cm. Also,

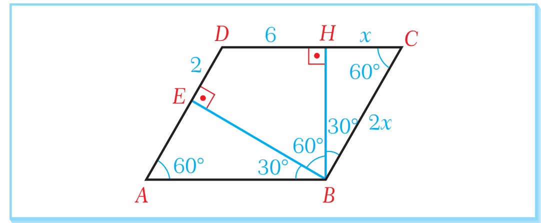
$$KO = KN + NO = 1 + 2 = 3 \text{ cm},$$

$$AO = AK + KO = 2KO = 6 \text{ cm}. \quad \text{(point } K \text{ is the midpoint of } AO)$$

So $AC = 2AO = 12$ cm.

EXAMPLE 23 $ABCD$ is a parallelogram and BH and BE are altitudes from vertex B to the sides DC and AD respectively. The measure of the angle between BH and BE is 60°, $DE = 2$ cm and $DH = 6$ cm. Find the lengths of sides AB and AD.

Solution We begin by drawing a figure with the information in the question:



$m(\angle EBH) = 60°$ and

$BE \perp AD$ and $BH \perp DC$ (since $AB \parallel DC$ and BH is an altitude). From the figure,

$$\begin{aligned} m(\angle ABE) &= m(\angle ABH) - m(\angle EBH) \\ &= 90° - 60° \\ &= 30°. \end{aligned}$$

In the right triangle ABE,

$$\begin{aligned} m(\angle A) + m(\angle ABE) + m(\angle BEA) &= 180° && \text{(sum of interior angles)} \\ m(\angle A) + 30° + 90° &= 180° \\ m(\angle A) &= 60° \\ m(\angle A) = m(\angle C) &= 60°. && \text{(opposite angles of a parallelogram)} \end{aligned}$$

In the right triangle CHB,

$$\begin{aligned} m(\angle C) + m(\angle CHB) + m(\angle HBC) &= 180° && \text{(sum of interior angles)} \\ 60° + 90° + m(\angle HBC) &= 180° \\ m(\angle HBC) &= 30°. \end{aligned}$$

Now let $HC = x$. Then

$BC = 2x$ (side opposite 30° is half of the hypotenuse)

$AB = DC = 6 + x$ (opposite sides of a parallelogram)

$AE = \dfrac{AB}{2} = \dfrac{6+x}{2}$ (side opposite 30° is half of the hypotenuse)

$AD = BC$ (opposite sides of a parallelogram)

$$2 + \frac{6+x}{2} = 2x$$

$$4 + 6 + x = 4x$$

$$3x = 10$$

$$x = \frac{10}{3} \text{ cm.}$$

So $AB = 6 + x = \dfrac{28}{3}$ cm and $BC = 2x = \dfrac{20}{3}$ cm.

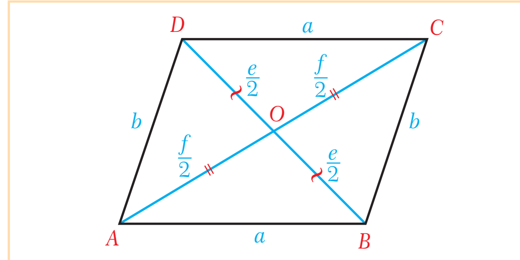
Remember:
In a 30° - 60° - 90° triangle, the length of the side opposite 30° is half the length of the hypotenuse.

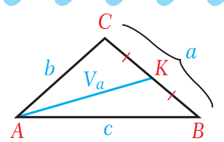
Theorem 9

In a parallelogram, the sum of the squares of the lengths of the diagonals is equal to twice the sum of the squares of the lengths of two consecutive sides.

Proof Let a, b and e, f be the lengths of the sides and diagonals of a parallelogram, respectively. Then we need to prove that

$$e^2 + f^2 = 2(a^2 + b^2).$$





In ΔABC,

$2V_a^2 + \frac{a^2}{2} = b^2 + c^2.$

Remember the theorem which relates the median of a triangle and its sides: if a, b and c are sides of a triangle and V_a is the median to side a, then $2V_a^2 + \frac{a^2}{2} = b^2 + c^2.$

In the figure above, $ABCD$ is a parallelogram.

Let us apply the median theorem to ΔDAC:

$$2 \cdot DO^2 + \frac{AC^2}{2} = AD^2 + DC^2$$

$$2 \cdot \left(\frac{e}{2}\right)^2 + \frac{f^2}{2} = b^2 + a^2$$

$$2 \cdot \frac{e^2}{4} + \frac{f^2}{2} = b^2 + a^2$$

$e^2 + f^2 = 2(b^2 + a^2)$. This is the required result.

EXAMPLE 24 The diagonals of a parallelogram measure 8 cm and $4\sqrt{6}$ cm, and the shorter side of the parallelogram measures half the length of its longer side. Find the perimeter of this parallelogram.

Solution Let x be the length of the shorter side and y be the length of the longer side of the parallelogram. Then from the question, $y = 2x$.

Let e and f be the lengths of the diagonals. Then

$$e^2 + f^2 = 2(a^2 + b^2) \qquad \text{(by Theorem 9)}$$

$$8^2 + (4\sqrt{6})^2 = 2 \cdot (x^2 + (2x)^2)$$

$$64 + 16 \cdot 6 = 10x^2$$

$$10x^2 = 160$$

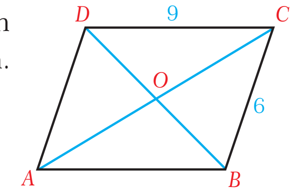
$$x^2 = 16$$

$$x = 4 \text{ cm.}$$

So the sides of the parallelogram measure 4 cm and 8 cm, and the perimeter of the parallelogram is $2 \cdot (4 + 8) = 24$ cm.

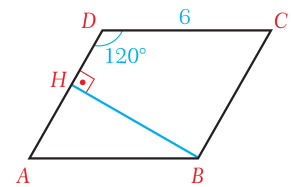
Check Yourself

1. In the figure, $ABCD$ is a parallelogram. Point O is the intersection point of diagonals AC and DB, and $DC = 9$ cm and $BC = 6$ cm. Given that $P(\Delta AOD) = 17$ cm, find $P(\Delta AOB)$.

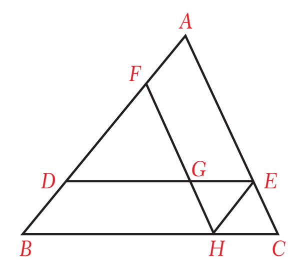


2. In the figure, $ABCD$ is a parallelogram.

   Find the length of the altitude BH if $DC = 6$ cm.



3. In the figure, ABC is a triangle and quadrilaterals $AFHE$ and $DBHE$ are parallelograms. If $DG = 2GE$ and $AB = 12$ cm, find the length of EH.



Answers

1. 20 cm 2. $3\sqrt{3}$ cm 3. 3 cm

3. Proving that a Quadrilateral Is a Parallelogram

As we have seen, if both pairs of opposite sides of a quadrilateral are parallel then by definition the quadrilateral is a parallelogram. Here are some more theorems which help us to prove that a quadrilateral is a parallelogram:

Theorem 10

If the diagonals of a quadrilateral bisect each other then the quadrilateral is a parallelogram.

Theorem 11

If both pairs of opposite sides of a quadrilateral are congruent then the quadrilateral is a parallelogram.

Theorem 12

If any two opposite sides of a quadrilateral are parallel and congruent then the quadrilateral is a parallelogram.

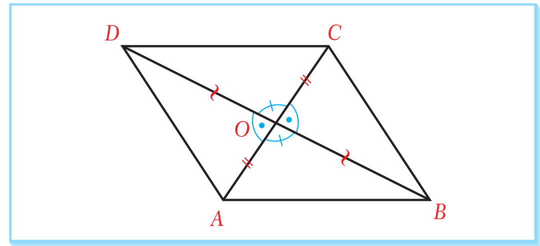
Theorem 13

If both pairs of opposite angles of a quadrilateral are congruent then the quadrilateral is a parallelogram.

We now have five ways to prove that a quadrilateral is a parallelogram: we can use the definition or one of the four theorems above.

EXAMPLE 25 Write a two-column proof of Theorem 10: if the diagonals of a quadrilateral bisect each other then the quadrilateral is a parallelogram.

Solution Look at the figure. Given that the diagonals of quadrilateral *ABCD* bisect each other, we need to prove that *ABCD* is a parallelogram. In other words, we need to show that both pairs of opposite sides of quadrilateral *ABCD* are parallel.

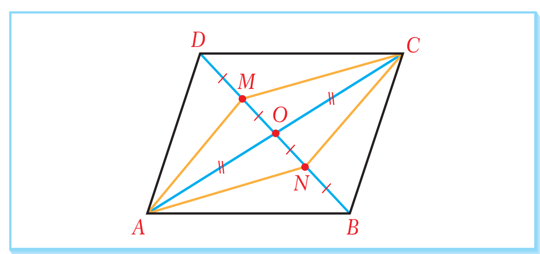


Statements	Reasons
1. $AO = OC$ and $BO = OD$	Given
2. $\angle BOC \cong \angle DOA$	Vertical angles
3. $\Delta BOC \cong \Delta DOA$	SAS postulate by 1 and 2
4. $\angle DOC \cong \angle BOA$	Vertical angles
5. $\angle OAD \cong \angle OCB$ and $\angle OBC \cong \angle ODA$	Corresponding angles of congruent triangles
6. $AD \parallel BC$	By 5
7. $\Delta BOA \cong \Delta DOC$	SAS postulate by 1 and 4
8. $\angle OAB \cong \angle OCD$ and $\angle OBA \cong \angle ODC$	Corresponding angles of congruent triangles
9. $AB \parallel DC$	By 8
10. $ABCD$ is a parallelogram.	By 6 and 9

Side Angle Side (SAS) postulate: If two sides and their interior angle in a triangle are congruent to two sides and their interior angle in another triangle, then the triangles are congruent.

EXAMPLE 26 In a parallelogram *ABCD*, point *O* is the intersection of diagonals *AC* and *BD*, and points *M* and *N* are midpoints of *DO* and *BO* respectively. Show that the quadrilateral *ANCM* is a parallelogram.

Solution Look at the figure. *ABCD* is a parallelogram. The diagonals of a parallelogram bisect each other, so $BO = OD$ and $AO = OC$.



M is the midpoint of *DO*, so $DM = MO = \dfrac{DO}{2}$.

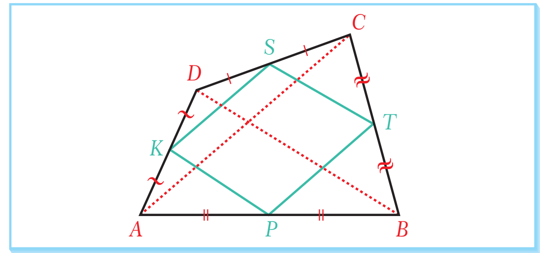
N is the midpoint of *BO*, so $BN = NO = \dfrac{BO}{2}$.

Since $DO = BO$ we have $MO = NO$.

MN and *AC* are diagonals of the quadrilateral *ANCM* and they bisect each other. So by Theorem 10, *ANCM* is a parallelogram.

EXAMPLE 27 Show that the quadrilateral which is formed by joining the midpoints of the sides of any quadrilateral is a parallelogram.

Solution Look at the figure. $ABCD$ is a quadrilateral, and points K, P, T and S are midpoints of the sides DA, AB, BC and CD respectively. We have to show that $KPTS$ is a parallelogram. In other words, we have to prove that both pairs of opposite sides of the quadrilateral $KPTS$ are parallel.

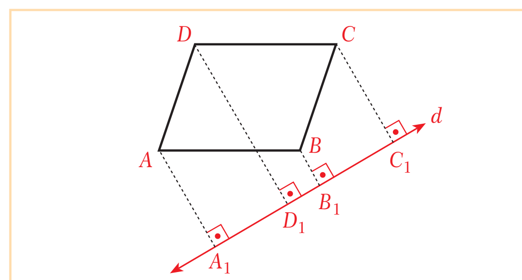


In ΔBDA, $KP \parallel BD$.	(KP is a midsegment)
In ΔBCD, $ST \parallel BD$.	(ST is a midsegment)
So $KP \parallel ST$.	(lines parallel to the same line are parallel)
In ΔDAC, $KS \parallel AC$.	(KS is a midsegment)
In ΔABC, $PT \parallel AC$.	(PT is a midsegment)
So $KS \parallel PT$.	(lines parallel to the same line are parallel)

So $KPTS$ is a parallelogram, since both pairs of its opposite sides are parallel.

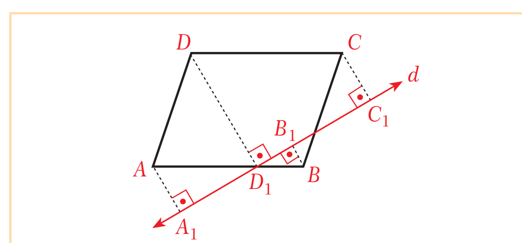
Theorem 14

Let $ABCD$ be a parallelogram, and let d be a line in the same plane. Then the following statements are true:



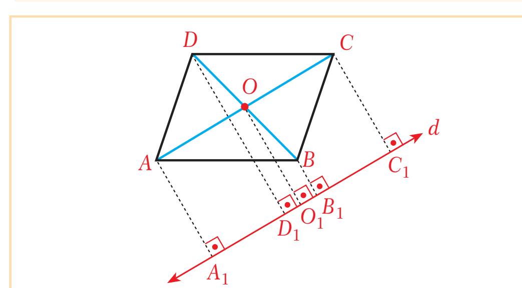
a. If line d is perpendicular to each of AA_1, BB_1, CC_1 and DD_1 as in the figure, then

$$AA_1 + CC_1 = DD_1 + BB_1.$$



b. If line d cuts the parallelogram $ABCD$ such that line d is perpendicular to each of AA_1, BB_1, CC_1 and DD_1 as in the figure, then

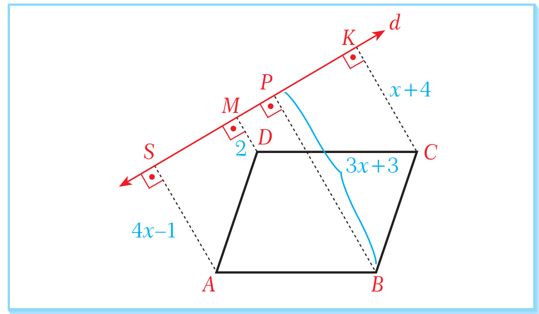
$$AA_1 + CC_1 = DD_1 - BB_1.$$



c. If point O is the intersection point of the diagonals of the parallelogram $ABCD$ and line d does not cut the parallelogram, and if line d is perpendicular to each of AA_1, BB_1, CC_1, DD_1 and OO_1 as in the figure, then

$$AA_1 + BB_1 + CC_1 + DD_1 = 4 \cdot OO_1.$$

EXAMPLE 28 The figure shows a parallelogram $ABCD$. Line d does not intersect $ABCD$ and d is perpendicular to each of AS, BP, CK and DM. Given $AS = 4x - 1$, $BP = 3x + 3$, $CK = x + 4$ and $MD = 2$ cm, find the value of x.



Solution By part **a** of Theorem 14 we can write

$$AS + CK = BP + MD.$$

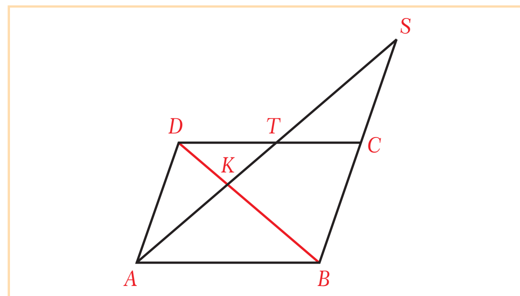
Substituting the given values into this equation gives us

$$4x - 1 + x + 4 = 3x + 3 + 2$$
$$2x = 2$$
$$x = 1 \text{ cm.}$$

Theorem 15

In the figure, $ABCD$ is a parallelogram. If points A, K, T, S and B, C, S are respectively collinear and if BD is a diagonal of the parallelogram, then

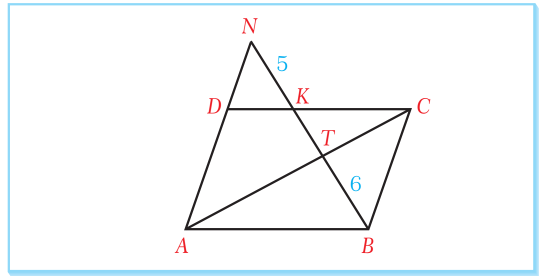
$$AK^2 = KT \cdot KS.$$



Proof We will use a two-column proof.

Statements	Reasons
1. $\angle KAB \cong \angle KTD$	Alternate interior angles
2. $\angle BKA \cong \angle DKT$	Vertical angles
3. $\Delta KAB \sim \Delta KTD$	AA similarity postulate
4. $\dfrac{KA}{KT} = \dfrac{KB}{KD}$	Corresponding sides of similar triangles are proportional.
5. $\angle DAK \cong \angle BSK$	Alternate interior angles
6. $\angle AKD \cong \angle SKB$	Vertical angles
7. $\Delta KBS \sim \Delta KDA$	AA similarity postulate
8. $\dfrac{KS}{KA} = \dfrac{KB}{KD}$	Corresponding sides of similar triangles are proportional.
9. $\dfrac{KA}{KT} = \dfrac{KS}{KA}$	By 4 and 8
10. $KA^2 = KS \cdot KT$	By 9

EXAMPLE 29 In the figure, $ABCD$ is a parallelogram, AC is its diagonal and points B, T, K, N and A, D, N are respectively collinear. If $BT = 6$ cm and $KN = 5$ cm, find the length of TK.



Solution By Theorem 15, $BT^2 = TK \cdot TN$.

Substituting the given values gives us the equality

$$6^2 = TK \cdot (TK + 5).$$

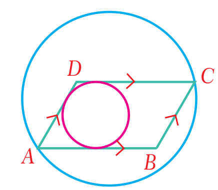
Let $TK = x$.

Then
$$6^2 = x \cdot (x + 5)$$
$$36 = x^2 + 5x$$
$$x^2 + 5x - 36 = 0$$
$$(x - 4)(x + 9) = 0 \qquad \text{(by factoring)}$$
$$x = 4 \text{ or } x = -9.$$

Since $x = -9$ is not a posssible length, $TK = 4$ cm.

Note

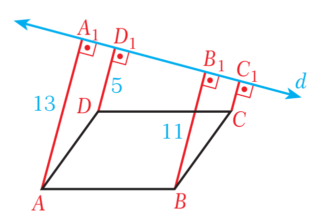
Since opposite angles of a parallelogram do not need to be supplementary and the sums of the lengths of opposite sides are not necessarily equal, a parallelogram cannot usually be inscribed or circumscribed.



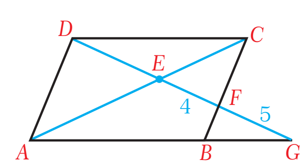
$ABCD$ cannot be inscribed or circumscribed

Check Yourself

1. In the figure opposite, line d does not intersect parallelogram $ABCD$ and d is perpendicular to each of AA_1, BB_1, CC_1 and DD_1. If $AA_1 = 13$ cm, $DD_1 = 5$ cm and $BB_1 = 11$ cm, find the length of CC_1.



2. In the figure opposite, $ABCD$ is a parallelogram. Points D, E, F and G are collinear, and point E is the intersection of DG and the diagonal AC. If $FG = 5$ cm and $EF = 4$ cm, find the length of DE.



Answers

1. 3 cm 2. 6 cm

C. RECTANGLE

1. Definition

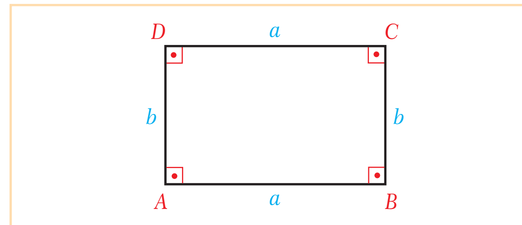
Definition

rectangle

A **rectangle** is a parallelogram which has four right angles.

A banknote is a common example of a rectangle.

We can also define a rectangle as a parallelogram with one right angle, since if one of the angles of a parallelogram is a right angle then the other three angles will also be right angles.



In the figure, $ABCD$ is a parallelogram with right angles $m(\angle A) = m(\angle B) = m(\angle C) = m(\angle D) = 90°$.

So $ABCD$ is a rectangle.

2. Properties of a Rectangle

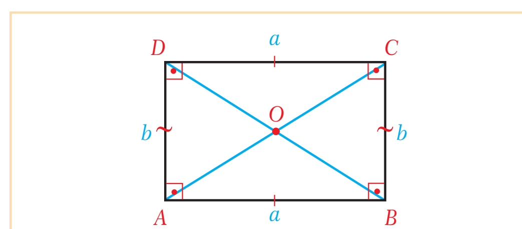
Since a rectangle is a type of parallelogram, it has all the properties of a parallelogram. It also has some additional properties.

Theorem 16

The diagonals of a rectangle are congruent.

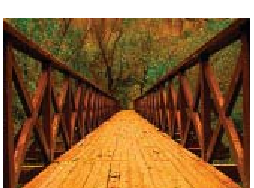
Proof

Look at the figure. Given that $ABCD$ is a rectangle, we need to prove $AC \cong BD$.



$ABCD$ is a rectangle, so it is a parallelogram.

$AD \cong BC$	(opposite sides of a parallelogram are congruent)
AB is a common side of ΔDAB and ΔCBA.	
$\angle DAB \cong \angle CBA$	(both right angles by definition of a rectangle)
$\Delta DAB \cong CBA$	(by SAS congruence postulate)
$AC \cong BD$	(corresponding sides of congruent triangles)

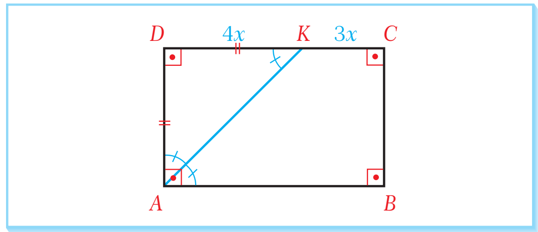


Moreover, since the rectangle is a parallelogram, its diagonals bisect each other: $AO \cong OC \cong BO \cong OD$.

So the diagonals of a rectangle are congruent and bisect each other. It can also be proven that if the diagonals of a parallelogram are the same length then this parallelogram is a rectangle.

EXAMPLE 30 The bisector of angle A of a rectangle $ABCD$ intersects side DC at a point K such that $DK : KC = 4 : 3$. Given that $DK = 16$ cm, find the lengths of all sides of $ABCD$ and its perimeter.

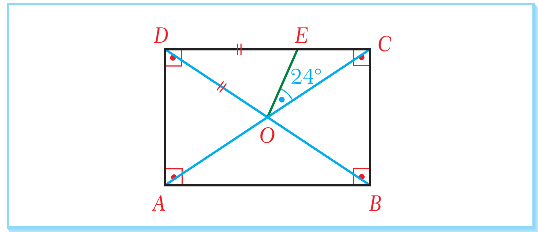
Solution Let x be the constant of proportionality. Since $DK : KC = 4 : 3$ we can write $DK = 4x$ and $KC = 3x$. Also, $DK = 16$ cm so $4x = 16$; $x = 4$ cm.



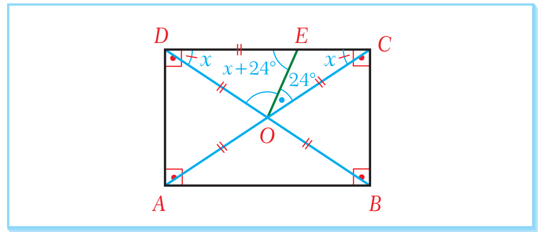
$AB \parallel DC$	(opposite sides of a rectangle are parallel)
$m(\angle BAK) = m(\angle DKA)$	(alternate interior angles)
ΔDAK is isosceles	(two congruent angles in ΔDAK)
$AD = DK = 16$ cm	(congruent legs of an isosceles triangle)
$DC = DK + KC = 7x = 28$ cm	
$AD = BC = 16$ cm	(opposite sides)
$DC = AB = 28$ cm	(opposite sides)

So the perimeter of $ABCD$ is $2 \cdot (16 + 28) = 88$ cm.

EXAMPLE 31 In the figure, $ABCD$ is a rectangle and point O is the intersection of diagonals AC and BD. Point E is on the side DC and $DO = DE$. Given $m(\angle EOC) = 24°$, calculate $m(\angle ODE)$.



Solution $ABCD$ is a rectangle. So the diagonals are equal and bisect each other. So $DO = OC$.



Let $m(\angle OCD) = x$, then

$m(\angle ODE) = m(\angle OCD) = x$	(base angles in isosceles triangle DOC)
$m(\angle DEO) = x + 24°$	(exterior angle of triangle OCE)
$m(\angle DEO) = m(\angle DOE) = x + 24°$	(base angles in isosceles triangle DOE)

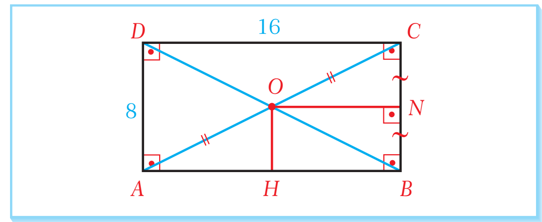
In triangle DOE,

$$m(\angle ODE) + m(\angle DEO) + (\angle DOE) = 180° \qquad \text{(sum of interior angles)}$$
$$x + x + 24° + x + 24° = 180°$$
$$3x = 132°$$
$$x = 44°.$$

So $m(\angle ODE) = 44°$.

EXAMPLE 32 A rectangle $ABCD$ has side lengths 8 cm and 16 cm, and point O is the intersection point of diagonals AC and BD. Find the distances from O to two consecutive sides of the rectangle.

Solution We begin by drawing the figure. The question asks us to find the lengths OH and ON.



Since $ON \perp BC$, it follows that $ON \parallel AB$. (lines perpendicular to the same line BC)

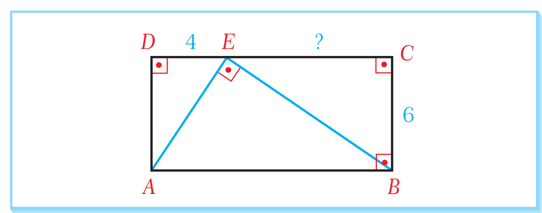
The diagonals bisect each other, so $AO = OC$. It follows that $BN = NC$.

So ON is a midsegment of ΔACB and $ON = \dfrac{AB}{2}$; $ON = 8$ cm.

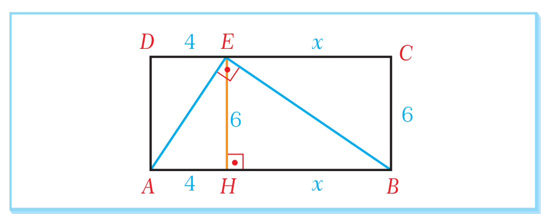
In a similar way we can show that OH is also a midsegment of ΔABC and $OH = \dfrac{BC}{2}$; $OH = 4$ cm.

So $ON = 8$ cm and $OH = 4$ cm.

EXAMPLE 33 In the figure, $ABCD$ is a rectangle and point E is on the side DC. Line segments AE and BE are perpendicular to each other. Given $DE = 4$ cm and $BC = 6$ cm, find the length of line segment EC.



Solution Let us draw a line EH which is perpendicular to side AB. Then $AHED$ and $HBCE$ are also rectangles.



Let $EC = x$, then $HB = x$. Also,

$DE = AH = 4$ cm (given), and

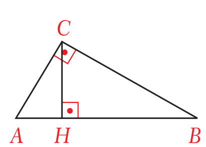
$BC = EH = 6$ cm (also given).

By the first Euclidean theorem,

$$EH^2 = AH \cdot HB$$
$$6^2 = 4 \cdot x$$
$$36 = 4x$$
$$x = 9 \text{ cm}.$$

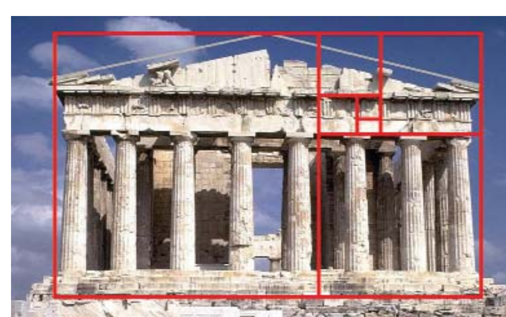
So $EC = 9$ cm.

First Euclidean theorem:



In ΔABC, if $m(\angle C) = 90°$ and $CH \perp AB$ then

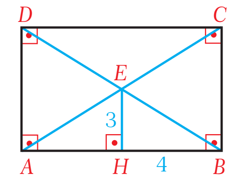
$CH^2 = AH \cdot HB$.



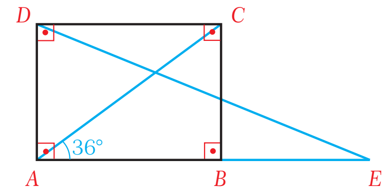
The Parthenon in Athens is an example of the architectural use of a shape known as the golden rectangle. The golden rectangle is thought to be the geometric form that is most pleasing to the human eye.

Check Yourself

1. In the figure, $ABCD$ is a rectangle and point E is the intersection of diagonals AC and BD. If $EH \perp AB$, $EH = 3$ cm and $HB = 4$ cm, find the length of AC.



2. The bisector of angle C of a rectangle $ABCD$ intersects side AD at point F such that $DF : FA = 3 : 2$. Find the perimeter of this rectangle if the length of side AB is 9 cm.
3. In the figure, $ABCD$ is a rectangle and points A, B and E are collinear. If $AC = BE$ and $m(\angle CAE) = 36°$, find $m(\angle AED)$.



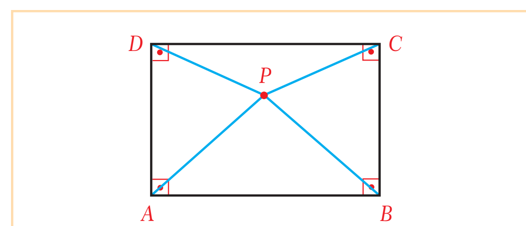
Answers

1. 10 cm 2. 48 cm 3. 18°

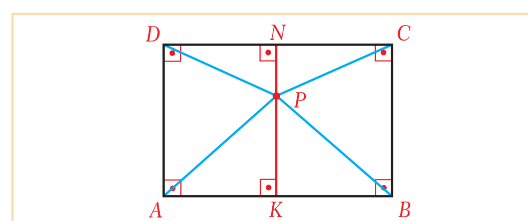
Theorem 17

In the figure, $ABCD$ is a rectangle. If P is any point in or on the rectangle then

$$PA^2 + PC^2 = PB^2 + PD^2.$$



Proof Let us draw NK through point P so that it is perpendicular to both sides AB and DC as in the figure.



In right triangles PAK and PKB,

$PK^2 = PA^2 - KA^2$ and $PK^2 = PB^2 - KB^2$. (Pythagorean Theorem)

So $PA^2 - KA^2 = PB^2 - KB^2$. (1)

In right triangles PNC and PND,

$PN^2 = PC^2 - NC^2$ and $PN^2 = PD^2 - ND^2$. (Pythagorean Theorem)

So $PC^2 - NC^2 = PD^2 - ND^2$. (2)

Adding equalities (1) and (2) side by side gives

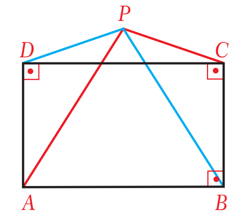
$$PA^2 - \cancel{KA^2} + PC^2 - \cancel{NC^2} = PB^2 - \cancel{KB^2} + PD^2 - \cancel{ND^2}, \qquad (KA = ND,\ KB = NC)$$

which means $PA^2 + PC^2 = PB^2 + PD^2$ as required.

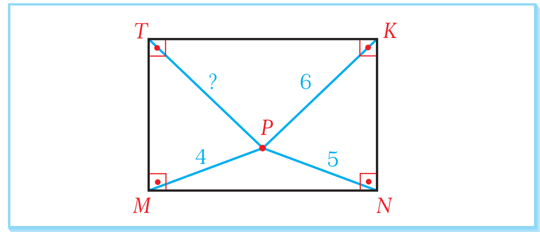
Note

Theorem 17 also holds if point P lies outside the rectangle: in the figure opposite,

$$PA^2 + PC^2 = PB^2 + PD^2.$$



EXAMPLE 34 In the figure, point P is an interior point of the rectangle $MNKT$. Given $PM = 4$ cm, $PN = 5$ cm and $PK = 6$ cm, find the length of line segment PT.

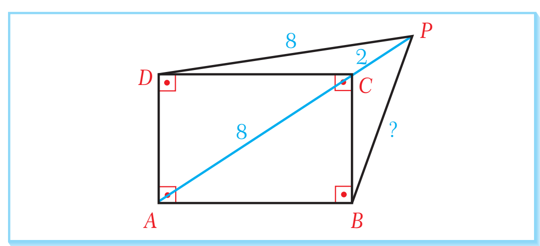


Solution By Theorem 17 we can write $PT^2 + PN^2 = PM^2 + PK^2$.

Let us substitute the given values in the equality:

$$PT^2 + 5^2 = 4^2 + 6^2$$
$$PT^2 + 25 = 16 + 36$$
$$PT^2 = 16 + 36 - 25$$
$$PT^2 = 27$$
$$PT = 3\sqrt{3} \text{ cm}.$$

EXAMPLE 35 In the figure, point P lies outside rectangle $ABCD$ and points A, C and P are collinear. If $PC = 2$ cm, $AC = 8$ cm and $PD = 8$ cm, find the length of line segment PB.



Solution By Theorem 17 we can write

$$PB^2 + PD^2 = PA^2 + PC^2.$$

Let us substitute the given values:

$$PB^2 + 8^2 = 10^2 + 2^2$$
$$PB^2 + 64 = 100 + 4$$
$$PB^2 = 104 - 64$$
$$PB^2 = 40$$
$$PB = 2\sqrt{10} \text{ cm}.$$

EXAMPLE 36 $ABCD$ is a rectangle and point E is on side DC with $DE < EC$. Point F is the midpoint of side DA. Given $FE \perp BE$, $FE = 12$ cm and $m(\angle EBA) = 30°$, find the perimeter of the rectangle.

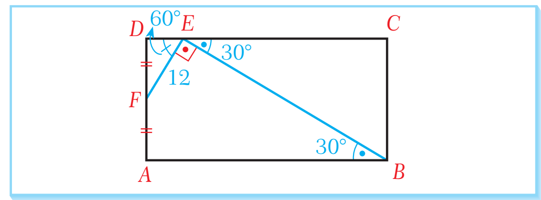
Solution Let us draw the figure. From it we can conclude

Drivers' licenses, credit cards and membership cards are all rectangular.

$$m(\angle ABE) = m(\angle BEC) = 30° \qquad (DC \parallel AB\text{, alternate interior angles})$$

$$m(\angle CED) = 180° \qquad \text{(straight angle)}$$

$$\begin{aligned} m(\angle FED) &= m(\angle CED) - m(\angle FEB) - m(\angle BEC) \\ &= 180° - 90° - 30° \\ &= 60°. \end{aligned}$$



In the right triangle EDF,

$$\sin 60° = \frac{DF}{FE} \qquad \text{(sine ratio)}$$

$$DF = \sin 60° \cdot FE$$

$$DF = \frac{\sqrt{3}}{2} \cdot 12 = 6\sqrt{3} \text{ cm},$$

$$\cos 60° = \frac{DE}{FE} \qquad \text{(cosine ratio)}$$

$$DE = \cos 60° \cdot FE$$

$$DE = \frac{1}{2} \cdot 12 = 6 \text{ cm}.$$

So $BC = DA = 2 \cdot DF = 12\sqrt{3}$ cm.

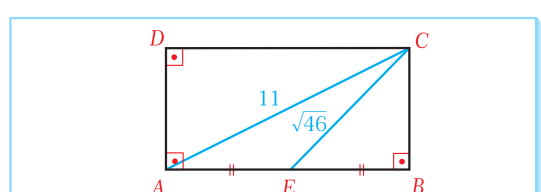
In the right triangle BCE,

$$\tan 30° = \frac{BC}{EC};\ EC = \frac{BC}{\tan 30°};\ EC = \frac{12\sqrt{3}}{\frac{1}{\sqrt{3}}};\ EC = 12\sqrt{3} \cdot \sqrt{3} = 36 \text{ cm, so}$$

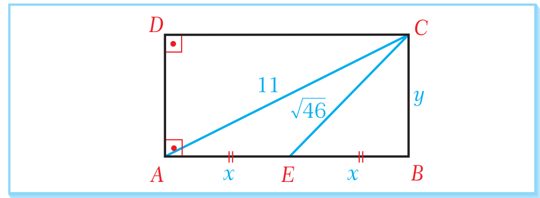
$$DC = EC + DE = 36 + 6 = 42 \text{ cm}.$$

So the perimeter of $ABCD$ is $2(12\sqrt{3} + 42) = (84 + 24\sqrt{3})$ cm.

EXAMPLE 37 In the figure, $ABCD$ is a rectangle and point E is the midpoint of side AB. AC is a diagonal of the rectangle, $AC = 11$ cm and $EC = \sqrt{46}$ cm. Find the lengths of the sides of the rectangle.



Solution Let $EB = x$ and $BC = y$.



So $AB = 2x$.

Also, $\angle B$ is a right angle.

In the right triangle BEC,

$$y^2 + x^2 = (\sqrt{46})^2;\ y^2 = 46 - x^2. \quad (1) \qquad \text{(Pythagorean Theorem)}$$

In the right triangle ABC,

$$y^2 + (2x)^2 = 11^2;\ y^2 + 4x^2 = 121. \quad (2) \qquad \text{(Pythagorean Theorem)}$$

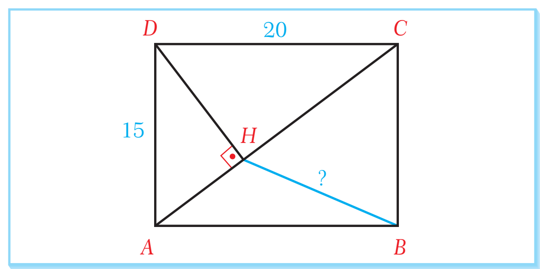
Substituting (1) in (2) gives

$$46 - x^2 + 4x^2 = 121$$
$$3x^2 = 75$$
$$x^2 = 25$$
$$x = 5, \text{ and}$$

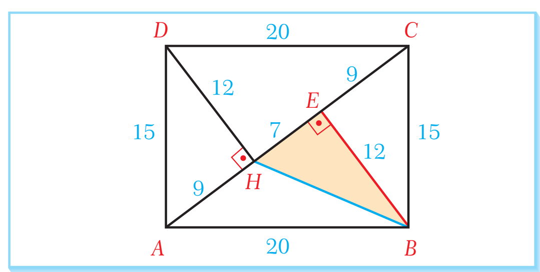
$$y^2 = 46 - x^2$$
$$y^2 = 21$$
$$y = \sqrt{21}.$$

So $AD = BC = \sqrt{21}$ cm and $DC = AB = 10$ cm.

EXAMPLE 38 In the figure, $ABCD$ is a rectangle. Point H is on the diagonal AC and DH is perpendicular to AC with $AD = 15$ cm and $DC = 20$ cm. Find the length of line segment HB.



Solution In the right triangle ADC,



$AC^2 = 15^2 + 20^2$ (Pythagorean Theorem)

$AC = 25$ cm

$AD^2 = AH \cdot AC$ (second Euclidean theorem)

$15^2 = AH \cdot 25$

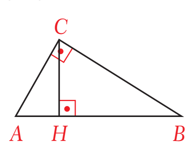
$AH = 9$ cm.

In the right triangle AHD,

$DH^2 = 15^2 - 9^2$ (Pythagorean Theorem)

$DH = 12.$

Second Euclidean theorem:



In ΔABC, if $m(\angle C) = 90°$ and $CH \perp AB$ then

$AC^2 = AH \cdot AB.$

Now let us construct BE such that $BE \perp AC$. Then we have

$$\angle DAH \cong \angle BCE \qquad \text{(alternate angles)}$$

$$\angle ADH \cong \angle CBE \qquad \text{(third angles in right triangles)}$$

$$BC \cong AD. \qquad \text{(opposite sides of a rectangle)}$$

So $\Delta AHD \cong \Delta CEB$, by the SAS congruence postulate.

So $AH = EC = 9$ cm and $DH = BE = 12$ cm, and

$$HE = AC - AH - EC$$

$$HE = 25 - 2 \cdot 9 = 7 \text{ cm.}$$

Finally, in the right triangle HEB,

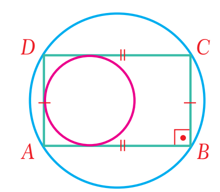
$$HB^2 = EB^2 + EH^2 \qquad \text{(Pythagorean Theorem)}$$

$$HB^2 = 12^2 + 7^2$$

$$HB = \sqrt{193} \text{ cm.}$$

Note

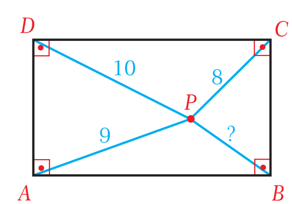
Since opposite angles of a rectangle are supplementary, we can always draw the circumscribed circle of a rectangle. However, it is not generally possible to construct its inscribed circle.



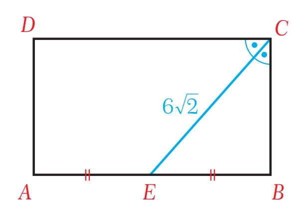
ABCD can be inscribed but not circumscribed.

Check Yourself

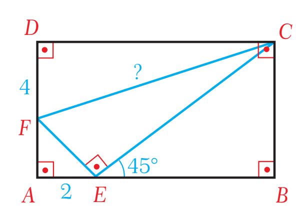
1. In the figure, $ABCD$ is a rectangle and P is a point in its interior. If $PA = 9$ cm, $PC = 8$ cm and $PD = 10$ cm, find the length of segment PB.



2. In the figure, $ABCD$ is a rectangle, point E is the midpoint of side AB and EC is the bisector of angle C. If $EC = 6\sqrt{2}$ cm, find $P(ABCD)$.



3. In the figure, $ABCD$ is a rectangle and points E and F are on the sides AB and DA respectively. Given $CE \perp FE$, $m(\angle CEB) = 45°$, $AE = 2$ cm and $DF = 4$ cm, find the length of segment CF.



Answers

1. $3\sqrt{5}$ cm 2. 36 cm 3. $4\sqrt{5}$ cm

D. RHOMBUS

1. Definition

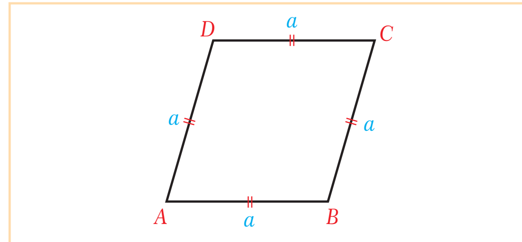
Definition

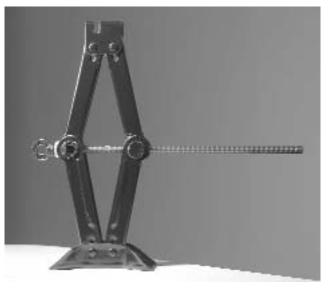
rhombus

A **rhombus** is a parallelogram whose sides are all congruent.

In the figure, *ABCD* is a parallelogram and $AB \cong BC \cong CD \cong DA$.

So *ABCD* is a rhombus.





Many objects that need to change in shape are built in the shape of a rhombus. The most useful property of a rhombus is that since the lengths of the sides are the same, opposite sides remain parallel as you change the measures of the angles. In addition, as you change the measures of the angles, the vertices slide along the lines of the diagonals and the diagonals remain perpendicular.

2. Properties of a Rhombus

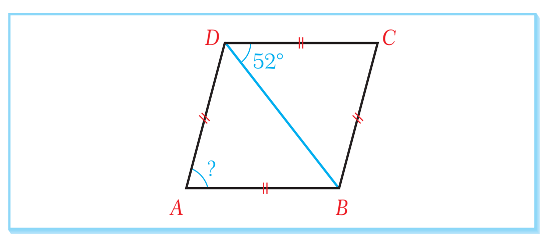
The plural form of rhombus is **rhombi**.

Since a rhombus is a parallelogram, it has all the properties of a parallelogram. It also has some additional properties that are not true for all parallelograms.

Theorem 18

Each diagonal of a rhombus bisects two angles of the rhombus.

EXAMPLE 39 In the figure, *ABCD* is a rhombus and $m(\angle CDB) = 52°$. Find $m(\angle DAB)$.



Solution We know that *ABCD* is a rhombus and that its diagonal bisects two angles (by Theorem 18).

So $m(\angle ADB) = m(\angle CDB) = 52°$

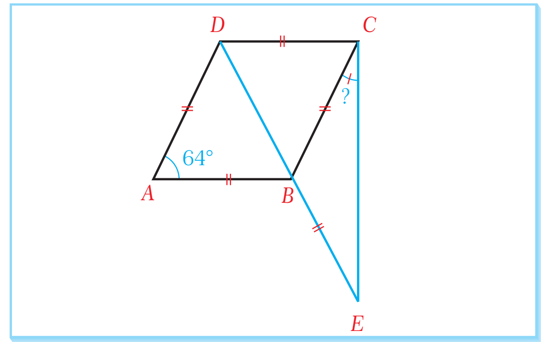
$m(\angle CDA) = 104°$

$m(\angle CDA) + m(\angle BAD) = 180°$ (consecutive angles in a parallelogram are supplementary)

$m(\angle DAB) = 180° - 104°$

$= 76°.$

EXAMPLE 40 In the figure, $ABCD$ is a rhombus, $AB = BE$ and points D, B and E are collinear. If $m(\angle A) = 64°$, find $m(\angle BCE)$.



Solution $ABCD$ is a rhombus and $AB = BC = BE$.

So ΔBEC is isosceles.

Let $m(\angle BCE) = x$. Then

$m(\angle BEC) = m(\angle BCE) = x$ (base angles in isosceles triangle BCE).

$m(\angle CBD) = 2x$ (exterior angle of ΔBEC)

$m(\angle DAB) + m(\angle ABC) = 180°$ (supplementary angles in a parallelogram)

$m(\angle ABC) = 180° - 64° = 116°$

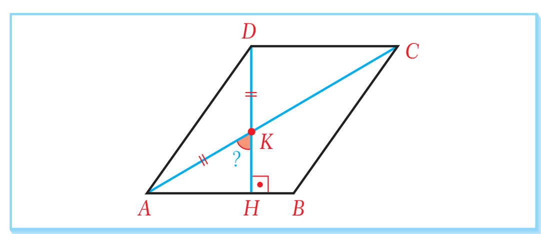
$$m(\angle CBD) = \frac{m(\angle ABC)}{2}$$ (diagonal BD is the bisector of $\angle ABC$)

$$2x = \frac{116°}{2};\ 2x = 58°;\ x = 29°.$$

So $m(\angle BCE) = 29°$.

Rhombi protect us.

EXAMPLE 41 In the figure, $ABCD$ is a rhombus, AC is its diagonal, DH is perpendicular to AB and $AK = KD$. Find $m(\angle AKH)$.



Solution $ABCD$ is a rhombus so its diagonal bisects its vertex angles. So $m(\angle HAK) = m(\angle DAK) = x$

and $m(\angle DAK) = m(\angle ADK) = x$ (base angles in isosceles triangle AKD)

$m(\angle AKH) = 2x$. (exterior angle of ΔAKH)

In ΔAKH,

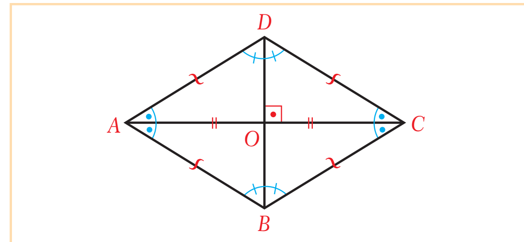
$$m(\angle HAK) + m(\angle AKH) + m(\angle AHK) = 180° \quad \text{(sum of interior angles)}$$
$$x + 2x + 90° = 180°$$
$$3x = 90°$$
$$x = 30°.$$

So $m(\angle AKH) = 2x = 60°$.

Theorem 19

The diagonals of a rhombus are perpendicular.

Proof Look at the figure. Since $ABCD$ is a rhombus, the diagonals bisect each other. So $AO = OC$ and $BO = OD$.



Triangles ABD, ABC, BCD and DAC are isosceles because the sides of a rhombus are congruent by definition. In an isosceles triangle, the median to the base is perpendicular to the base and also bisects the vertex angle. So

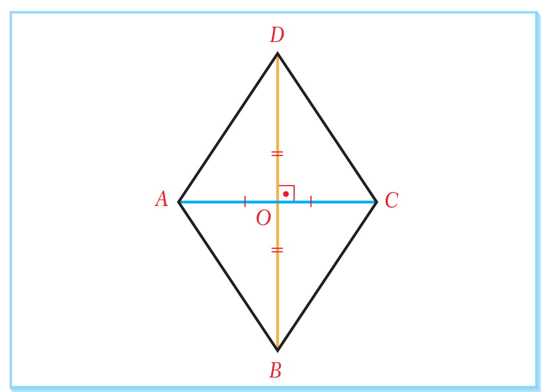
$$AO \perp BD \text{ and } m(\angle BAO) = m(\angle OAD)$$
$$BO \perp AC \text{ and } m(\angle ABO) = m(\angle OBC)$$
$$CO \perp BD \text{ and } m(\angle BCO) = m(\angle OCD)$$
$$DO \perp AC \text{ and } m(\angle ADO) = m(\angle ODC).$$

So AC and DB are the bisectors of each pair of vertex angles, and also the diagonals are perpendicular to each other.

It can also be shown that if either the diagonals of a parallelogram are perpendicular to each other, or if one of the diagonals bisects two angles of the parallelogram, then the parallelogram is a rhombus.

Theorem 20

In a rhombus, the sum of the squares of the lengths of the diagonals is equal to four times the square of the length of one side.



Proof In the figure, $ABCD$ is a rhombus. AC and BD are the diagonals and point O is the intersection of the diagonals.

We need to prove that $DB^2 + AC^2 = 4 \cdot AD^2$.

Since the diagonals bisect each other, $DO = OB$ and $AO = OC$.

So $DO = \dfrac{DB}{2}$ and $AO = \dfrac{AC}{2}$.

In ΔAOD, $\angle O$ is a right angle. (diagonals are perpendicular)

So $AO^2 + DO^2 = AD^2$ (Pythagorean Theorem)

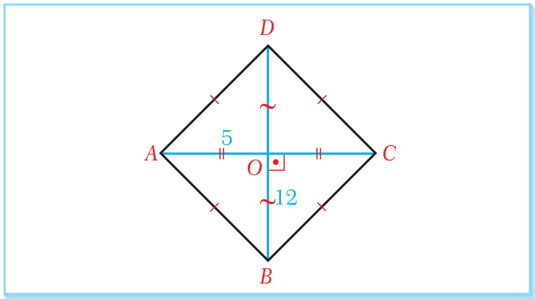
$$\left(\frac{DB}{2}\right)^2 + \left(\frac{AC}{2}\right)^2 = AD^2$$

$\dfrac{DB^2}{4} + \dfrac{AC^2}{4} = AD^2$. Multiplying both sides by 4 gives us

$DB^2 + AC^2 = 4 \cdot AD^2$, which is the desired result.

EXAMPLE 42 Find the perimeter of a rhombus whose diagonals measure 10 cm and 24 cm.

Solution 1 Let $ABCD$ be the rhombus. AC and BD are the diagonals, $AC = 10$ cm and $BD = 24$ cm.



By Theorem 20, we have

$$BD^2 + AC^2 = 4AB^2$$
$$10^2 + 24^2 = 4AB^2$$
$$4AB^2 = 676$$
$$AB^2 = 169$$
$$AB = 13 \text{ cm.}$$

So the perimeter of the rhombus is $4 \cdot AB = 4 \cdot 13 = 52$ cm.

Solution 2 $AO = \dfrac{AC}{2}$ and $BO = \dfrac{BD}{2}$ (diagonals bisect each other)

So $AO = 5$ cm and $BO = 12$ cm.

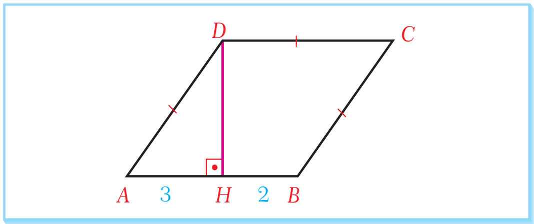
Also, $m(\angle AOB) = 90°$. (diagonals are perpendicular)

In the right triangle AOB,

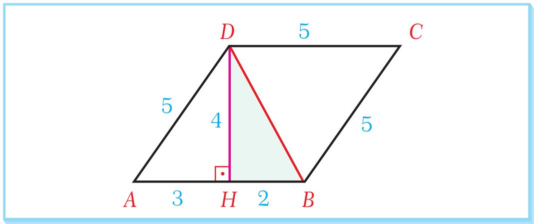
$$AB^2 = AO^2 + OB^2 \quad \text{(Pythagorean Theorem)}$$
$$AB^2 = 5^2 + 12^2$$
$$AB^2 = 169$$
$$AB = 13 \text{ cm.}$$

So the perimeter of the rhombus is $4 \cdot AB = 4 \cdot 13 = 52$ cm.

EXAMPLE 43 In the figure, $ABCD$ is a rhombus, DH is perpendicular to AB and $AH = 3$ cm, $HB = 2$ cm. Find the lengths of the diagonals of this rhombus.



Solution $ABCD$ is a rhombus, so its sides are congruent:



$$AD = AB = AH + HB$$
$$AD = 3 + 2 = 5 \text{ cm.}$$

In the right triangle AHD,

$$DH^2 = AD^2 - AH^2 \quad \text{(Pythagorean Theorem)}$$
$$DH^2 = 5^2 - 3^2$$
$$DH^2 = 16$$
$$DH = 4 \text{ cm.}$$

Let us construct the diagonal BD. Then in the right triangle DHB,

$$DB^2 = DH^2 + HB^2$$
$$DB^2 = 4^2 + 2^2$$
$$DB^2 = 20$$
$$DB = 2\sqrt{5} \text{ cm}.$$

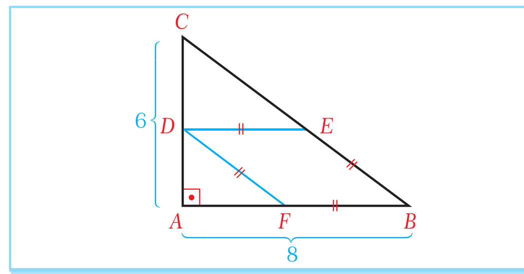
By Theorem 20, $DB^2 + AC^2 = 4AD^2$

$$20 + AC^2 = 4 \cdot 5^2$$
$$AC^2 = 80$$
$$AC = 4\sqrt{5} \text{ cm}.$$

So the diagonals measure $2\sqrt{5}$ cm and $4\sqrt{5}$ cm.

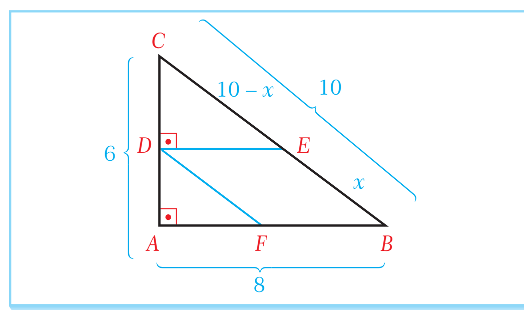
The arms of the lifting platform shown in the picture make rhombus shapes. When the lift is operating, the lengths of the diagonals change but the lengths of the sides do not change. Can you imagine how this lift would look and work if its arms formed parallelograms that were not rhombi?

EXAMPLE 44 In the figure, ΔABC is a right triangle and $DFBE$ is a rhombus. Given that $AB = 8$ cm and $AC = 6$ cm, find the length of one side of the rhombus.



Solution In the right triangle ABC,

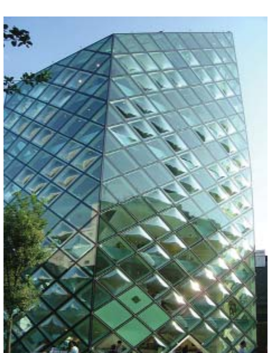
$$BC^2 = AB^2 + AC^2 \text{ (Pythagorean Theorem)}$$
$$BC^2 = 8^2 + 6^2$$
$$BC^2 = 100$$
$$BC = 10.$$



Let the length of one side of the rhombus be x, so $EB = x$ and $CE = 10 - x$. Then

$DE \parallel FB$ (opposite sides of the rhombus)

$m(\angle CDE) = m(\angle A)$. (corresponding angles)



$m(\angle C)$ is also a common angle of ΔBCA and ΔECD, so $\Delta CDE \sim \Delta CAB$ by the AA similarity postulate. Therefore,

$$\frac{CD}{CA} = \frac{DE}{AB} = \frac{CE}{CB}; \ \frac{x}{8} = \frac{10-x}{10} \quad \text{(lengths of corresponding sides are proportional)}$$

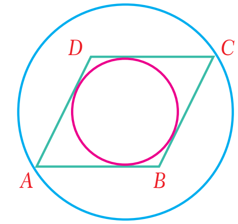
$$10x = 80 - 8x$$

$$18x = 80; \ x = \frac{40}{9} \text{ cm}.$$

So one side of the rhombus measures $\frac{40}{9}$ cm.

Note

Since opposite angles of a rhombus are not supplementary it is not possible to construct the circumscribed circle of a rhombus, but it is possible to construct the inscribed circle of a rhombus. In the figure, $ABCD$ is a rhombus.



ABCD can be circumscribed but not inscribed.

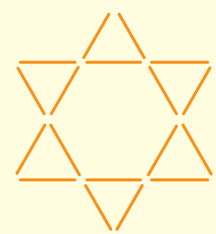
Activity

There are many simple things you can do to improve your creative thinking ability. Everyone knows that solving puzzles is a good way to develop your creative thinking and problem solving skills. These skills are not just useful for math: they can help you understand the world around you, too.

Here are two puzzles that you can try to solve using matchsticks or toothpicks. Good luck, and enjoy!

1. Move two matchsticks in the pattern to make one rhombus and one equilateral triangle.

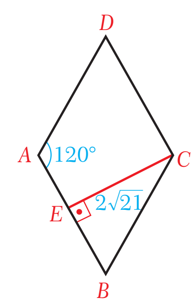
2. Move six of the matchsticks below to make a new figure made up of six congruent rhombi.



Check Yourself

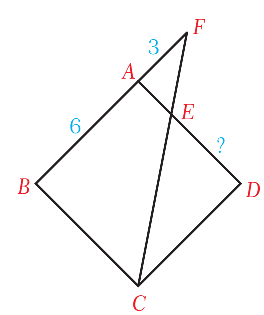
1. $ABCD$ is a rhombus and point E is on side DC such that $m(\angle BEC) = 55°$. If $m(\angle A) = 100°$, find $m(\angle DBE)$.

2. In the figure, $ABCD$ is a rhombus, E is a point on side AB and CE is perpendicular to side AB. If $m(\angle A) = 120°$ and $EC = 2\sqrt{21}$, find the perimeter of the rhombus.

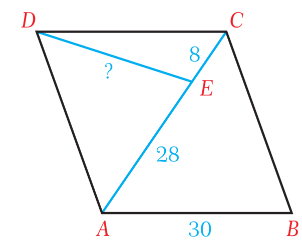


This lamp stays perpendicular to the wall as it moves into the room. Can you explain why, using your knowledge of rhombi?

3. In the figure, $ABCD$ is a rhombus and points C, E, F and B, A, F are respectively collinear. $BA = 6$ cm and $AF = 3$ cm are given. Find the length of the line segment ED.



4. In the figure, $ABCD$ is a rhombus and point E is on the diagonal AC. $EC = 8$ cm, $AE = 28$ cm and $AB = 30$ cm are given. Find the length of DE.



Answers

1. 15° 2. $16\sqrt{7}$ cm 3. 4 cm 4. 26 cm

E. SQUARE

1. Definition

Definition

square

A **square** is a rectangle whose sides are all congruent.

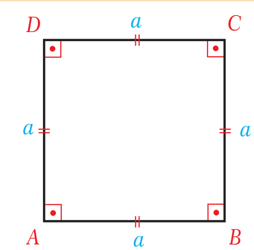
A chessboard is a square board which is divided into smaller squares of two contrasting colors. Do you know how many squares there are on a chessboard?

In the figure, $ABCD$ is a square since it is a rectangle and all the sides are congruent:

$AB = BC = CD = DA = a$.

We can also define a square as a rhombus with four right angles. In $ABCD$,

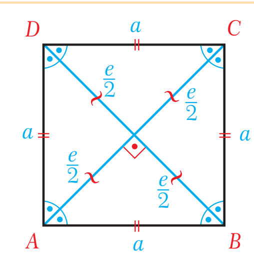
$m(\angle A) = m(\angle B) = m(\angle C) = m(\angle D) = 90°$.



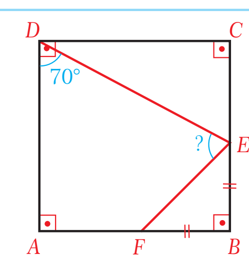
2. Properties of a Square

We can say that a square is both a rectangle and a rhombus. So it has all the properties of a square and a rhombus, i.e.:

1. Its diagonals have the same length.
2. Its diagonals are perpendicular.
3. Its diagonals bisect each other.
4. Each diagonal bisects two interior angles.



EXAMPLE 45 In the figure, $ABCD$ is a square and points E and F are on the sides BC and AB respectively. FB is congruent to BE and $m(\angle ADE) = 70°$. Find $m(\angle DEF)$.



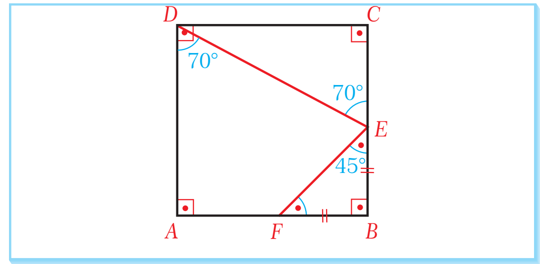
Solution $m(\angle ADE) = m(\angle CED) = 70°$ ($AD \parallel BC$, alternate interior angles)

Also, $m(\angle BFE) = m(\angle BEF)$ and (base angles in isosceles triangle BEF)

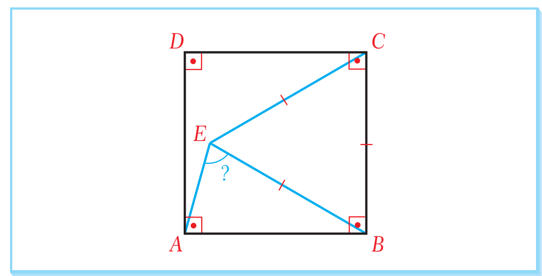
$m(\angle BEF) = \frac{90°}{2}$; $m(\angle BEF) = 45°$

$m(\angle BEC) = 180°$ (straight angle)

$$m(\angle DEF) = 180° - m(\angle CED) - m(\angle BEF)$$
$$= 180° - 70° - 45°$$
$$= 65°.$$



EXAMPLE 46 In the figure, $ABCD$ is a square and ΔBEC is equilateral. Find the measure of $\angle AEB$.



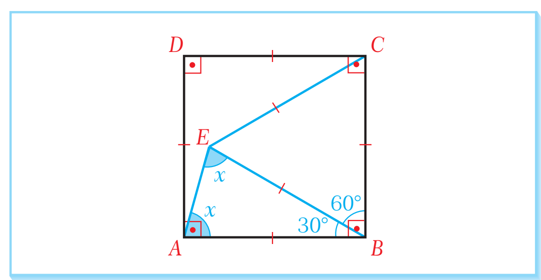
Solution $AB = BC$ ($ABCD$ is a square)

$BC = BE$ (ΔBEC is equilateral)

So $AB = BE$ and ΔABE is isosceles.

Also, $m(\angle EBC) = 60°$ (equilateral triangle)

$$m(\angle ABE) = m(\angle B) - m(\angle EBC)$$
$$= 90° - 60°$$
$$= 30°.$$



In ΔABE,

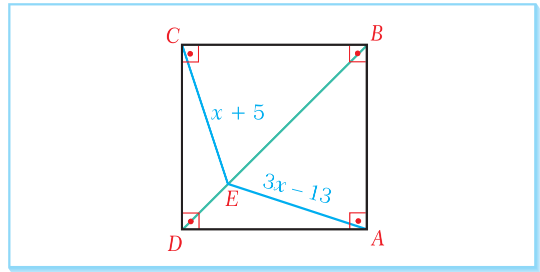
$m(\angle BAE) = m(\angle BEA) = x$ (base angles in ΔABE)

$m(\angle BAE) + m(\angle AEB) + m(\angle ABE) = 180°$ (sum of interior angles)

$$x + x + 30° = 180°$$
$$2x = 150°$$
$$x = 75°.$$

So $m(\angle AEB) = 75°$.

EXAMPLE 47 In the figure, $ABCD$ is a square and point E is on the diagonal DB such that $CE = x + 5$ and $EA = 3x - 13$. What is the value of x?

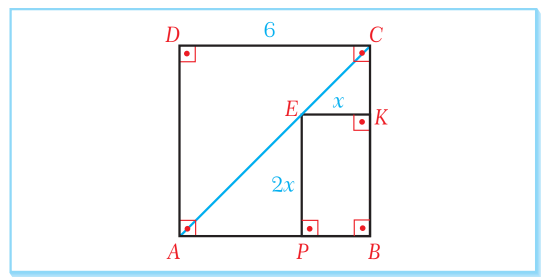


Solution

1. $CD \cong DA$ (sides of a square are congruent)
2. $DE \cong DE$ (common side of ΔCDE and ΔADE)
3. $\angle CDE \cong \angle ADE$ (diagonal DB is the bisector of $\angle D$)
4. $\Delta CDE \cong \Delta ADE$ (by SAS congruence postulate)
5. $CE \cong AE$ (corresponding sides of congruent triangles)
6. $CE = AE$ (congruent sides have equal lengths)
7. $x + 5 = 3x - 13$

$$2x = 18$$
$$x = 9 \text{ cm}$$

EXAMPLE 48 In the figure, $ABCD$ is a square and $PBKE$ is a rectangle. Point E is on the diagonal AC. If $DC = 6$ cm and the length of EK is half of the length of EP, find the length of EK.



Solution AC bisects $\angle DAB$ because it is the diagonal of a square. So $m(\angle CAD) = 45°$.

In ΔAPE, $m(\angle APE) = 90°$ and

$m(\angle PEA) = 180° - (90° + 45°)$ (sum of interior angles)

$m(\angle PEA) = 45°$.

So $AP = PE$.

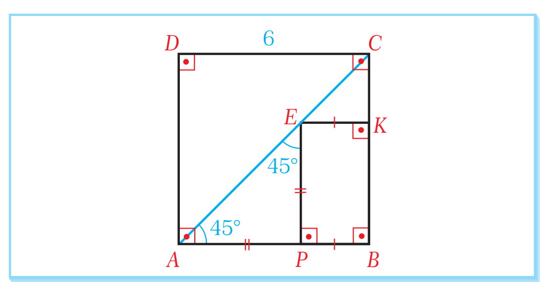
Also, $EK = PB$ (opposite sides of a rectangle)

and $AP = 2 \cdot EK$. (given)

$AB = AP + PB$; $AB = 3 \cdot EK$

$AB = DC$; $3 \cdot EK = 6$ cm

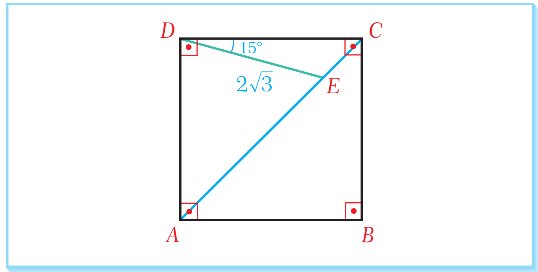
So $EK = 2$ cm.



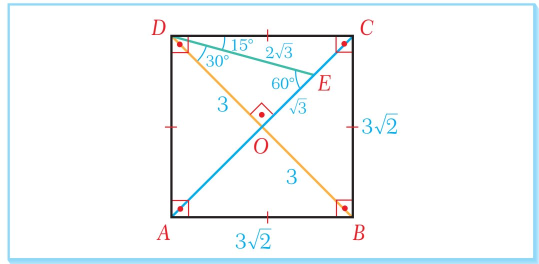
Property 7

If the length of one side of a square is b then length of its diagonal is $b\sqrt{2}$.

EXAMPLE 49 In the figure, $ABCD$ is a square and point E is on the diagonal AC such that $m(\angle CDE) = 15°$. Find the perimeter of the square if $DE = 2\sqrt{3}$ cm.



Solution Let us draw the diagonal DB, so $DB \perp AC$.



Point O is the intersection of diagonals, and

$$\begin{aligned} m(\angle EDO) &= m(\angle CDO) - m(\angle CDE) \\ &= 45° - 15° \\ &= 30°. \end{aligned}$$

In the right triangle DOE,

$$\cos 30° = \frac{DO}{DE};\; DO = DE \cdot \cos 30° = 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 3 \text{ cm}.$$

Also, $DO = OB$ (diagonals bisect each other)

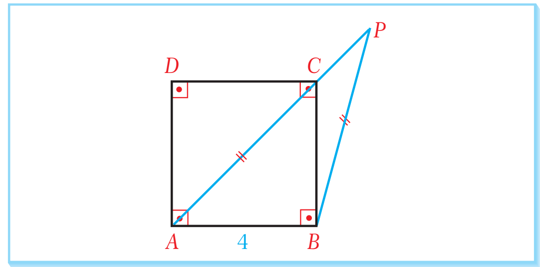
$$DB = DO + OB = 6 \text{ cm}.$$

In a square, since the length of the diagonal is $\sqrt{2}$ times the length of one side, we get

$$DB = \sqrt{2} \cdot AB$$

$$AB = \frac{DB}{\sqrt{2}} = \frac{6}{\sqrt{2}} = 3\sqrt{2} \text{ cm}.$$

EXAMPLE 50 In the figure, $ABCD$ is a square. Points A, C and P are collinear, $AC = BP$ and the length of one side of the square is 4 cm. Find the length of line segment CP.



Solution Let us draw the diagonal DB, so $DB \perp AC$. Point O is the intersection of the diagonals. So

$BD = AC = BP = 4\sqrt{2}$

$OB = OD = OC = OA = 2\sqrt{2}$ (diagonals bisect each other)

ΔPOB is a right triangle. (diagonals are perpendicular)

In ΔPOB,

$PO^2 + OB^2 = PB^2$ (Pythagorean Theorem)

$PO^2 + (2\sqrt{2})^2 = (4\sqrt{2})^2$

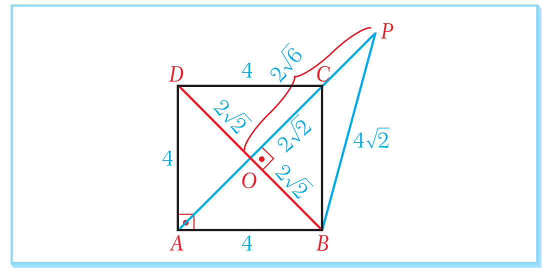
$PO^2 + 8 = 32$

$PO^2 = 24;\ \ PO = 2\sqrt{6}$ cm.

Finally, $PC = PO - CO$

$= 2\sqrt{6} - 2\sqrt{2}$

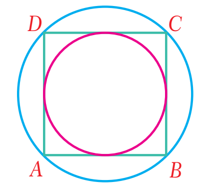
$= 2(\sqrt{6} - \sqrt{2})$ cm.



How many squares?
How many circles?
How many triangles?

Note

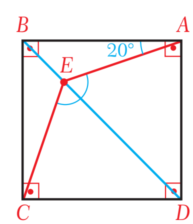
Since opposite angles in a square are supplementary and the sum of lengths of opposite sides is equal to the sum of the lengths of the other two opposite sides, a square is both an inscribed and circumscribed quadrilateral. In the figure, $ABCD$ is a square.



$ABCD$ is inscribed and circumscribed.

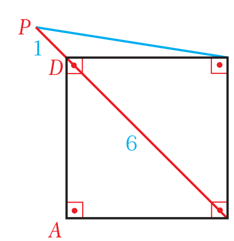
Check Yourself 14

1. In the figure, $ABCD$ is a square and BD is a diagonal of the square. Point E is on BD and $m(\angle BAE) = 20°$. Find $m(\angle AEC)$.



2. AC is a diagonal of a square $ABCD$. Points E and F are on the sides AC and AB respectively, and FE is perpendicular to AC. Find the length of EC if $AF = 4\sqrt{2}$ cm and $FB = 2\sqrt{2}$ cm.

3. In the figure, $ABCD$ is a square and points P, D and B are collinear. If $PD = 1$ cm and $BD = 6$ cm, find the length of the line segment PC.

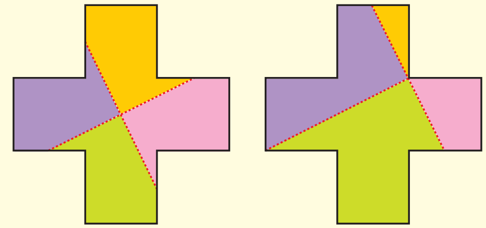


Answers

1. 130° 2. 8 cm 3. 5 cm

Activity

Copy the shapes opposite onto a piece of paper and cut them out. Cut along the dotted lines to make four pieces from each shape. Then try to make a quadrilateral from each set of four pieces.



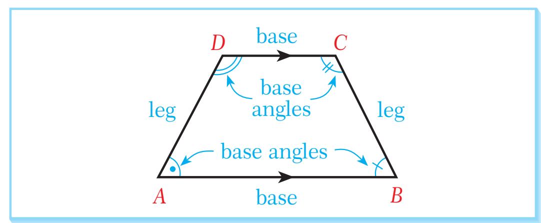
F. TRAPEZOID

1. Definition

Definition

trapezoid, base, leg, base angles, altitude, height

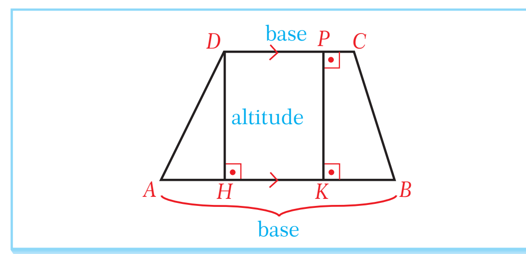
A **trazepoid** is a quadrilateral which has exactly one pair of parallel sides.



The parallel sides of the trapezoid are called the **bases** of the trapezoid. The other sides are the **legs**.

Two angles that share a base of the trapezoid are called **base angles**.

In the top figure at the right, $ABCD$ is a quadrilateral, $DC \parallel AB$ and AD is not parallel to BC. So by the definition, $ABCD$ is a trapezoid. Sides DC and AB are the bases, and sides AD and BC are the legs.



A perpendicular line segment drawn from any point on one of the bases to any point on the other base is called an **altitude** of the trapezoid. The length of any altitude is called the **height** of the trapezoid.

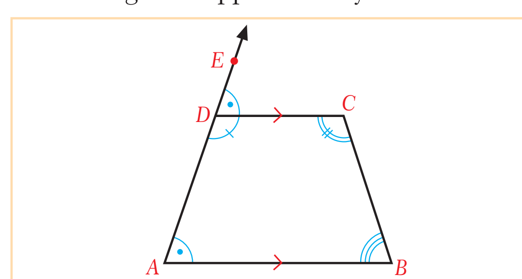
In the figure opposite, DH and PK are two altitudes of the trapezoid.

2. Properties of a Trapezoid

Theorem 21

In a trapezoid, two interior angles that share the same leg are supplementary.

Proof



In the figure, $ABCD$ is a trapezoid with $DC \parallel AB$.

We need to prove that

$m(\angle A) + m(\angle D) = 180°$ and

$m(\angle B) + m(\angle C) = 180°$.

If we extend AD so that points A, D and E are collinear, we get

$m(\angle A) = m(\angle EDC)$ (corresponding angles)

$m(\angle D) + m(\angle EDC) = 180°.$ (supplementary angles)

So $m(\angle D) + m(\angle A) = 180°$, as required.

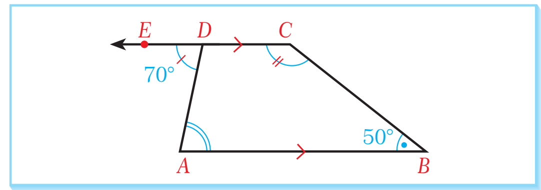
In a similar way, we can prove that $m(\angle B) + m(\angle C) = 180°$.

EXAMPLE 51

In the figure, $ABCD$ is a trapezoid, points C, D and E are collinear, and $DC \parallel AB$.

$m(\angle ABC) = 50°$ and $m(\angle ADE) = 70°$ are given.

Find the measures of all the interior angles of the trapezoid.



Solution

$m(\angle ABC) + m(\angle BCD) = 180°$ (by Theorem 21)

$50° + m(\angle BCD) = 180°$

$m(\angle BCD) = 180° - 50°$

$m(\angle BCD) = 130°$

$m(\angle DAB) = m(\angle ADE) = 70°$ (alternate interior angles)

$m(\angle ADC) + m(\angle ADE) = 180°$ (supplementary angles)

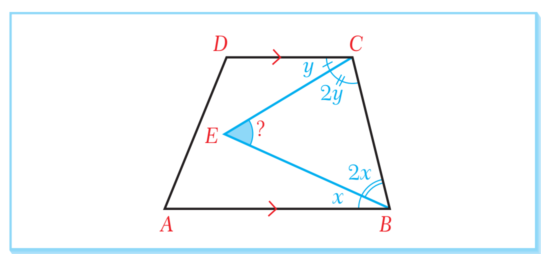
$m(\angle ADC) = 180° - 70°$

$m(\angle ADC) = 110°$

So $m(\angle B) = 50°$, $m(\angle C) = 130°$, $m(\angle A) = 70°$ and $m(\angle D) = 110°$.

EXAMPLE 52

In the figure, $ABCD$ is a trapezoid. Point E lies inside the trapezoid and $DC \parallel AB$. Given $m(\angle EBC) = 2 \cdot m(\angle EBA) = 2x$ and $m(\angle ECB) = 2 \cdot m(\angle ECD) = 2y$, find $m(\angle CEB)$.



Solution

$m(\angle B) + m(\angle C) = 180°$ (two angles that share the same leg are supplementary)

$3x + 3y = 180°$

$x + y = 60°$

In ΔCEB,

$m(\angle CEB) + 2x + 2y = 180°$ (sum of the measures of interior angles)

$m(\angle CEB) + 2(x + y) = 180°$

$m(\angle CEB) = 180° - 2 \cdot 60°$

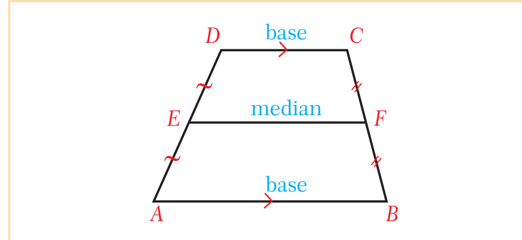
$m(\angle CEB) = 60°.$

Definition

median of a trapezoid

The **median** of a trapezoid is the line segment that joins the midpoints of the legs.

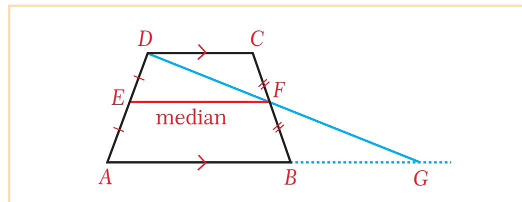
In the figure, points E and F are midpoints of the legs AD and BC respectively. So line segment EF is the median of trapezoid $ABCD$.



Theorem 22

The median of a trapezoid is parallel to the bases and its length is half of the sum of the lengths of the bases.

Proof In the figure, $ABCD$ is a trapezoid, $DC \parallel AB$ and EF is the median of the trapezoid.



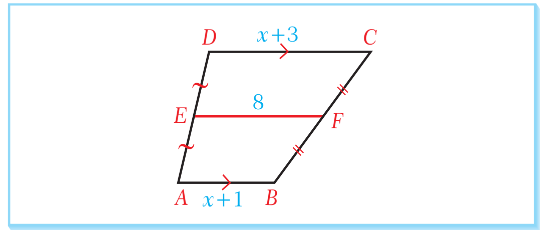
We have to prove that $EF \parallel AB \parallel DC$ and $EF = \dfrac{AB+CD}{2}$.

Let us begin by drawing DF to intersect line AB at point G, and continue with a two-column proof:

Statements	Reasons
1. $\angle DFC \cong \angle GFB$	Vertical angles
2. $\angle DCB \cong \angle GBF$	Alternate interior angles
3. $CF = FB$	Definition of median
4. $\Delta DFC \cong \Delta GFB$	ASA congruence postulate by 1, 2 and 3
5. $DF = GF$	Corresponding sides of equal triangles
6. $DC = BG$	Corresponding sides of equal triangles
7. $AG = AB + BG$	Addition of line segments
8. $AG = AB + DC$	By 6 and 7
9. $AE = ED$	Definition of median
10. EF is the midline of ΔADG	By 5 and 9
11. $EF \parallel AG$ and $EF = \dfrac{AG}{2}$	By 10
12. $EF \parallel AB \parallel DC$ and $EF = \dfrac{AB+CD}{2}$	Bases are parallel, and combining 8 and 11

EXAMPLE 53 In trapezoid $ABCD$ in the figure, $DC \parallel AB$ and EF is the median of the trapezoid.

$EF = 8$ cm, $AB = x + 1$ and $DC = x + 3$ are given. Find the lengths of AB and DC.



Solution $EF = \dfrac{AB+CD}{2}$ since EF is the median. So

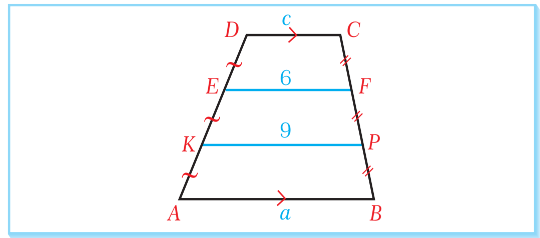
$$8 = \frac{x+1+x+3}{2}$$

$$2x + 4 = 16; \quad 2x = 12; \quad x = 6 \text{ cm.}$$

So $AB = 7$ cm and $DC = 10$ cm.

EXAMPLE 54 In trapezoid $ABCD$ in the figure, $AK = KE = ED$ and $BP = PF = FC$.

Given $AB = a$, $DC = c$, $KP = 9$ cm and $EF = 6$ cm, find a and c.



Solution Since $AK = KE = ED$ and $BP = PF = FC$, by Thales' theorem we obtain $DC \parallel EF \parallel KP \parallel AB$. So quadrilaterals $ABFE$ and $KPCD$ are trapezoids.

In trapezoid $ABFE$,

$$KP = \frac{AB+EF}{2}; \quad 9 = \frac{a+6}{2} \qquad (KP \text{ is the median of the trapezoid})$$

$$a + 6 = 18$$

$$a = 12 \text{ cm.}$$

In trapezoid $KPCD$,

$$EF = \frac{KP+DC}{2}; \quad 6 = \frac{c+9}{2} \qquad (EF \text{ is the median of the trapezoid})$$

$$c + 9 = 12$$

$$c = 3 \text{ cm.}$$

So $a = 12$ cm and $c = 3$ cm.

Thales' theorem of parallel lines: If three or more parallel lines intersect two transversals, then they divide the transversals proportionally.

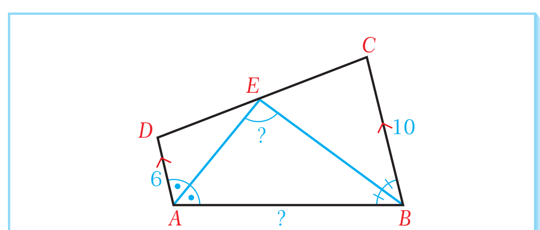
EXAMPLE 55 In trapezoid $ABCD$ in the figure, $AD \parallel BC$ and AE and BE are the bisectors of angles A and B respectively.

Given $AD = 6$ cm and $BC = 10$ cm,

a. find $m(\angle AEB)$.

b. show that $DE = EC$.

c. find the length of AB.



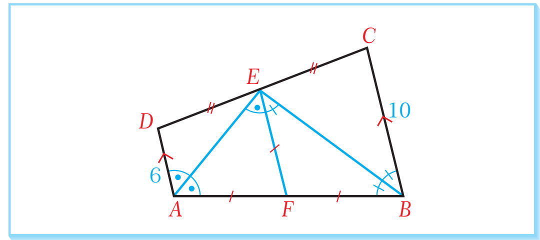
Solution **a.** $m(\angle A) + m(\angle B) = 180°$

(supplementary angles)

$$m(\angle EAB) = \frac{m(\angle A)}{2},\ m(\angle EBA) = \frac{m(\angle B)}{2}$$

$$m(\angle EAB) + m(\angle EBA) = \frac{m(\angle A) + m(\angle B)}{2}$$

$$= \frac{180°}{2} = 90°$$



In ΔAEB,

$m(\angle EAB) + m(\angle EBA) + m(\angle AEB) = 180°$ (sum of the measures of interior angles)

$90° + m(\angle AEB) = 180°$

$m(\angle AEB) = 90°$.

b. Let us draw line segment EF parallel to the bases: $EF \parallel DA \parallel CB$. Then

$m(\angle CBE) = m(\angle FEB)$ (alternate interior angles)

$EF = FB$ (congruent angles in ΔEFB)

$m(\angle DAE) = m(\angle AEF)$ (alternate interior angles)

$AF = EF$. (congruent angles in ΔAFE)

So $EF = AF = FB$.

Since $EF \parallel AD \parallel BC$ and EF bisects AB, by Thales' theorem of parallel lines we can conclude that EF bisects side DC.

So $DE = EC$, and EF is the median of the trapezoid.

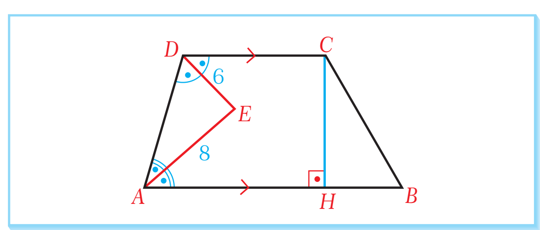
c. Since EF is the median, $EF = \dfrac{AD + BC}{2}$

$$= \frac{6 + 10}{2} = 8 \text{ cm.}$$

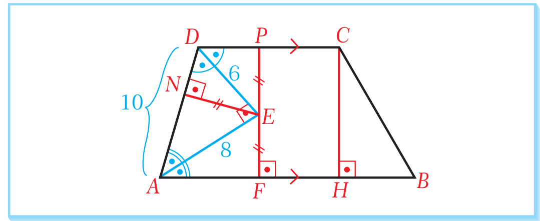
ΔAEB is a right triangle and EF is the median to the hypotenuse of the triangle. In a right triangle, the length of the median to the hypotenuse is half of the length of hypotenuse. So $AB = 16$ cm.

EXAMPLE 56 In the figure, $ABCD$ is a trapezoid, $AB \parallel DC$, and AE and DE are the bisectors of $\angle A$ and $\angle D$ respectively.

If $CH \perp AB$, $AE = 8$ cm and $DE = 6$ cm, find the height CH.



Solution Since $AB \parallel DC$, we know from Example 83 that $m(\angle AED) = 90°$.



In ΔAED,

$$AD^2 = DE^2 + AE^2 \text{ (Pythagorean Theorem)}$$
$$AD^2 = 6^2 + 8^2$$
$$AD^2 = 100; \quad AD = 10 \text{ cm.}$$

Remember:
Any point on the bisector of an angle is equidistant from the two sides of the angle.

Let us draw the perpendiculars EP, EF and EN so

$NE = EF$ (AE is a bisector)

$NE = EP$. (DE is a bisector)

So $PE = EF$ and points P, E and F are collinear.

Also, $CH = PF = PE + EF$; $CH = 2 \cdot PE$.

In ΔAED,

$AD \cdot NE = DE \cdot AE$ (Euclidean theorem)

$$10 \cdot NE = 6 \cdot 8$$
$$NE = \frac{24}{5}$$

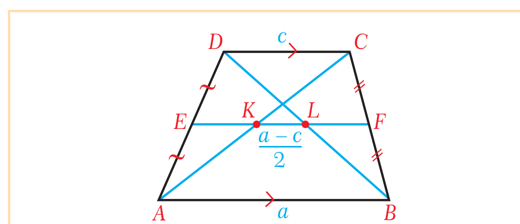
So $CH = 2 \cdot EP = 2 \cdot NE = \dfrac{48}{5}$ cm. ($NE = EP = EF$)

Theorem 23

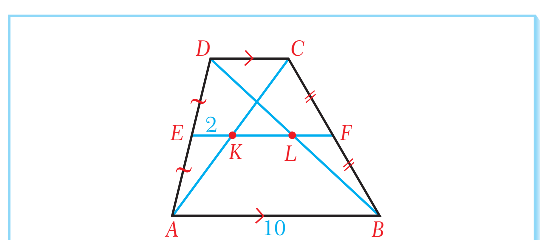
The length of the segment of the median of a trapezoid which lies between the diagonals of the trapezoid is half the difference of the lengths of the bases.

In the figure, EF is the median of trapezoid $ABCD$. AC and BD are diagonals of the trapezoid and they intersect median EF at points K and L. So by Theorem 23,

$$KL = \frac{a-c}{2}.$$



EXAMPLE 57 In trapezoid $ABCD$ in the figure, EF is the median and AC and BD are diagonals. Given that $EK = 2$ cm and $AB = 10$ cm, find the lengths of DC and KL.



Solution 1 $EF \parallel AB \parallel DC$ (EF is the median of the trapezoid)

Since $EF \parallel AB$ and point E is the midpoint of AD, then by the triangle proportionality theorem, point K is the midpoint of AC.

So EK is the midsegment of ΔACD, and

$$EK = \frac{DC}{2}$$
$$DC = 2 \cdot EK$$
$$DC = 2 \cdot 2 = 4 \text{ cm}$$
$$KL = \frac{AB - DC}{2} \quad \text{(by Theorem 23)}$$
$$KL = \frac{10 - 4}{2}$$
$$KL = 3 \text{ cm.}$$

So $DC = 4$ cm and $KL = 3$ cm.

Solution 2 EL is the midline of ΔABD. So $EL = \frac{AB}{2} = 5$ cm.

Also, $KL = EL - EK$, so $KL = 5 - 2 = 3$ cm.

Since EK is the midsegment of ΔACD, $DC = 2 \cdot EK$. So $DC = 4$ cm and $KL = 3$ cm.

EXAMPLE 58 $ABCD$ is a trapezoid with $AB \parallel CD$, $m(\angle D) = 130°$, $m(\angle B) = 65°$, $AD = 10$ cm and $DC = 5$ cm. Find the length of AB.

Solution We begin by drawing the figure, then draw CK parallel to AD, intersecting side AB at point K as shown below. Then quadrilateral $ADCK$ is a parallelogram, since $DC \parallel AK$ and $AD \parallel CK$. Also,

$DC = AK = 5$ cm and $AD = KC = 10$ cm (opposite sides of a parallelogram)

$m(\angle D) = m(\angle CKA) = 130°$ (opposite angles of a parallelogram)

$m(\angle CKA) = m(\angle B) + m(\angle BCK)$ (exterior angle property of a triangle)

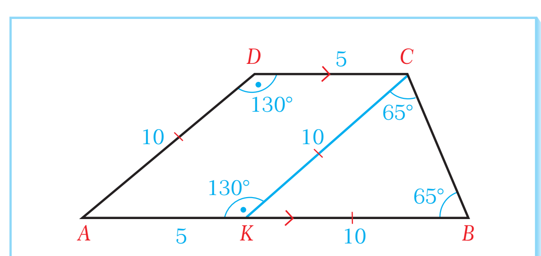
$m(\angle BCK) = 130° - 65°$

$= 65°.$

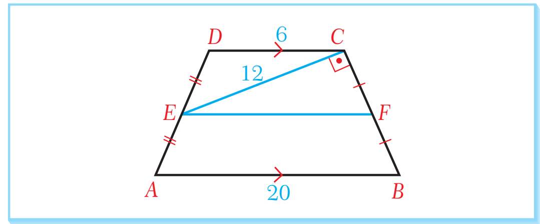
ΔKBC is isosceles, so $KC = KB = 10$ cm. ($m(\angle BCK) = m(\angle B)$)

$AB = AK + KB = 5 + 10$

$= 15$ cm.



EXAMPLE 59 In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$ and point E is the midpoint of AD. Given $AB = 20$ cm, $DC = 6$ cm and $EC = 12$ cm, find the length of BC.



Solution Let us draw EF parallel to the bases and intersecting side BC at point F, so $EF \parallel AB \parallel DC$.

Then EF is the median of the trapezoid,

$$CF = FB \text{ and } EF = \frac{AB + DC}{2}$$

$$= \frac{20+6}{2} = 13 \text{ cm.}$$

In the right triangle ECF,

$$CF^2 = EF^2 - EC^2 \qquad \text{(Pythagorean Theorem)}$$

$$= 13^2 - 12^2$$

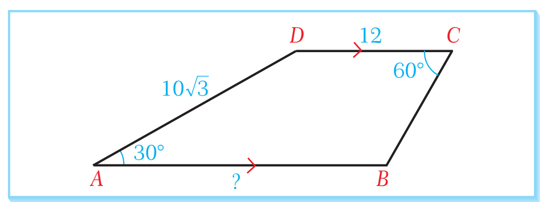
$$= 169 - 144$$

$$= 25$$

$$CF = 5 \text{ cm.}$$

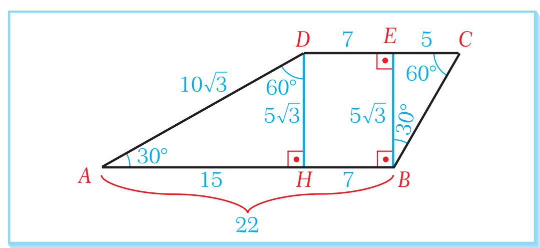
So $CB = 2 \cdot CF = 10$ cm.

EXAMPLE 60 In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$. $AD = 10\sqrt{3}$ cm, $DC = 12$ cm, $m(\angle A) = 30°$ and $m(\angle C) = 60°$ are given. Find the length of AB.



Solution Let us draw DH and BE such that $DH \perp AB$ and $BE \perp DC$, as shown in the figure. Then

$DH = BE$. (altitudes of a trapezoid)



In the right triangle AHD,

$$\cos 30° = \frac{AH}{AD}; \quad AH = AD \cdot \cos 30°$$

$$= 10\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 15 \text{ cm, and}$$

$$\sin 30° = \frac{DH}{AD}; \quad DH = AD \cdot \sin 30°$$

$$= 10\sqrt{3} \cdot \frac{1}{2} = 5\sqrt{3} \text{ cm.}$$

So $EB = DH = 5\sqrt{3}$ cm.

In the right triangle CEB,

$$\tan 60° = \frac{EB}{EC}; \quad EC = \frac{EB}{\tan 60°}$$

$$= \frac{5\sqrt{3}}{\sqrt{3}} = 5 \text{ cm.}$$

Finally,

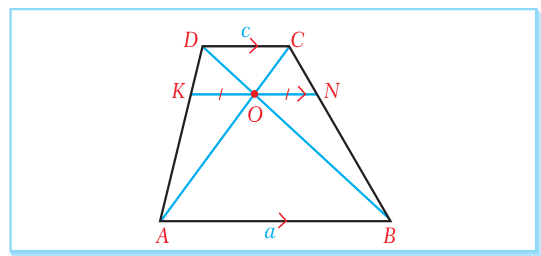
$DE = DC - EC = 12 - 5 = 7$ cm and $HB = DE = 7$ cm. (opposite sides of rectangle $HBED$)

Also, $AB = AH + HB = 15 + 7 = 22$ cm, so $AB = 22$ cm.

EXAMPLE 61 In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$. Point O is the intersection point of the diagonals AC and BD, and $AB = a$ and $DC = c$. If KN is parallel to the bases then prove that

a. $KO = ON$.

b. $KN = \dfrac{2 \cdot a \cdot c}{a + c}$.



Solution a. $\Delta AKO \sim \Delta ADC$ by the AA similarity postulate.

Corresponding sides of similar triangles are proportional, so

$$\frac{AK}{AD} = \frac{KO}{DC}. \qquad (1)$$

Similarly, $\Delta DKO \sim \Delta DAB$ by the AA similarity postulate, so

$$\frac{DK}{AD} = \frac{KO}{AB}. \qquad (2)$$

Adding equations (1) and (2) side by side gives

$$\frac{AK + DK}{AD} = \frac{KO}{DC} + \frac{KO}{AB}$$

$$1 = KO\left(\frac{1}{DC} + \frac{1}{AB}\right) \quad (\text{since } AK + DK = AD)$$

$$\frac{1}{KO} = \frac{1}{DC} + \frac{1}{AB}. \qquad (3)$$

In a similar way, by using the similarities $\Delta BON \sim \Delta BDC$ and $\Delta CON \sim \Delta CAB$ we obtain the equation

$$\frac{1}{ON} = \frac{1}{DC} + \frac{1}{AB}. \qquad (4)$$

Since the right sides of equations (3) and (4) are equal, the left sides are equal, too.

So $KO = ON$, as required.

b. From equation (3) in part **a** we have

$$\frac{1}{KO} = \frac{1}{DC} + \frac{1}{AB}; \quad \frac{1}{KO} = \frac{1}{c} + \frac{1}{a} \qquad \text{(finding the common denominator)}$$

$$\frac{1}{KO} = \frac{a+c}{a \cdot c}$$

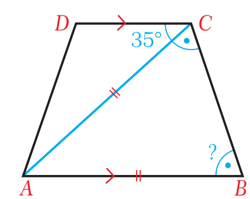
$$KO = \frac{a \cdot c}{a+c}.$$

Since $KN = KO + ON$ we have $KN = 2 \cdot KO$ (from part a).

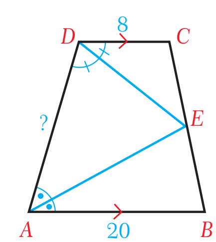
So $KN = \dfrac{2 \cdot a \cdot c}{a+c}$, as required.

Check Yourself 15

1. In the figure, $ABCD$ is a trapezoid with $DC \parallel AB$.
   Given $m(\angle DCA) = 35°$ and $AC = AB$, find $m(\angle B)$.

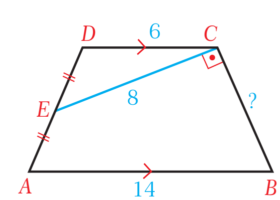


2. In the figure, point E is on side BC of a trapezoid $ABCD$ with $AB \parallel DC$. AE and DE are bisectors of angles $\angle A$ and $\angle D$ respectively, $AB = 20$ cm and $DC = 8$ cm. Find the length of AD.



3. $ABCD$ is a trapezoid with bases AB and CD. Given $AB = 15$ cm, $BC = 6$ cm, $CD = 5$ cm and $DA = 8$ cm, find the height of the trapezoid.

4. $ABCD$ in the figure is a trapezoid with $AB \parallel DC$. Point E is the midpoint of AD, $AB = 14$ cm, $DC = 6$ cm and $EC = 8$ cm. Find the length of BC.



Answers

1. $72.5°$ 2. 28 cm 3. $\dfrac{24}{5}$ cm 4. 12 cm

3. Isosceles Trapezoids

a. Definition

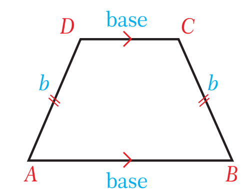
Definition

isosceles trapezoid

An **isosceles trapezoid** is a trapezoid whose legs are congruent.

In the figure, $AB \parallel DC$ and $AD = BC$.

So $ABCD$ is an isosceles trapezoid.



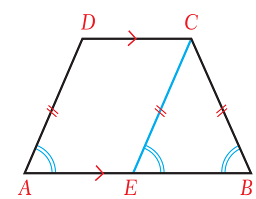
b. Properties of an isosceles trapezoid

An isosceles trapezoid has all the properties of a regular trapezoid. It also has some additional properties. Let us look at them in turn.

Theorem 24

The base angles of an isosceles trapezoid are congruent.

Proof In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$ and $AD = BC$.



We have to prove $\angle A \cong \angle B$ and $\angle C \cong \angle D$.

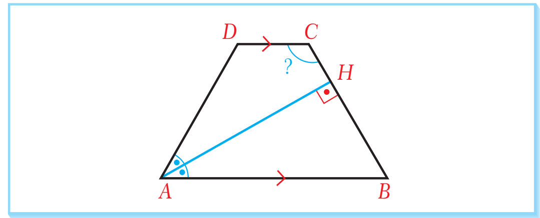
Let us draw CE so that $CE \parallel DA$ and point E is on AB. Then

1. $ADCE$ is a parallelogram ($AB \parallel DC$ and $CE \parallel AD$)
2. $AD = CE$. (opposite sides of a parallelogram)
3. Since $AD = BC$, we have $AD = BC = CE$ and
4. $\angle CEB \cong \angle B$ ($BC = CE$)
5. $\angle A \cong \angle CEB$ (corresponding angles)
6. $\angle A \cong \angle B$.
7. Since two interior angles that share the same leg are supplementary, it follows that
   $m(\angle A) + m(\angle D) = 180°$ and
   $m(\angle B) + m(\angle C) = 180°$, i.e.
   $m(\angle D) = m(\angle C)$ ($m(\angle A) = m(\angle B)$)
8. $\angle D \cong \angle C$, which completes the proof.

It can also be shown that if the base angles in a trapezoid are congruent then the trapezoid is an isosceles trapezoid.

EXAMPLE 62 In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$ and $AH \perp BC$.

If AH is the bisector of $\angle A$, find $m(\angle C)$.



Solution $m(\angle A) = m(\angle B)$ (base angles of an isosceles trapezoid)

In the right triangle ABH,

$$\frac{m(\angle A)}{2} + m(\angle B) + 90° = 180° \quad \text{(sum of interior angles of a triangle)}$$

$$\frac{m(\angle A)}{2} + m(\angle A) + 90° = 180°$$

$$m(\angle A) = 60°.$$

So $m(\angle A) = m(\angle B) = 60°$

$$m(\angle B) + m(\angle C) = 180°$$

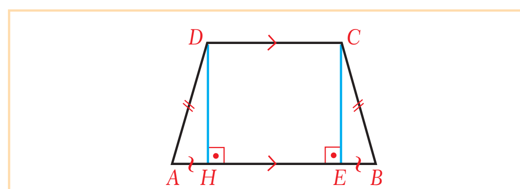
$$m(\angle C) = 180° - 60°$$

$$= 120°.$$

Theorem 25

The perpendicular projections of the legs of an isosceles trapezoid are congruent and the length of each leg equals half the difference of the lengths of the bases.

Proof In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$ and $AB > DC$. DH and CE are two altitudes and AH and EB are projections of the legs AD and BC respectively.



We need to show that $AH = EB = \dfrac{AB - DC}{2}$.

1. $\Delta ADH \cong \Delta BCE$ (SAS congruence postulate)
2. $AH \cong EB$ (corresponding sides of congruent triangles)
3. $DC = HE$ (opposite sides of a rectangle)
4. $AH + EB = AB - HE$

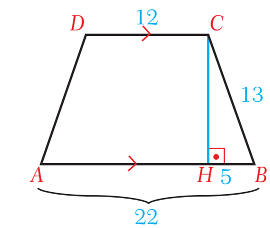
$$2 \cdot AH = AB - DC$$

$$AH = \frac{AB - DC}{2}$$

So $AH = EB = \dfrac{AB - DC}{2}$.

EXAMPLE 63 $ABCD$ is an isosceles trapezoid with $AB \parallel DC$. Given $AB = 22$ cm, $DC = 12$ cm and $BC = 13$ cm, find the height of the trapezoid.

Solution Let us draw the altitude $CH \perp AB$ as shown in the figure.



Then $HB = \dfrac{AB - CD}{2}$ by Theorem 25, i.e.

$$HB = \frac{22 - 12}{2} = 5 \text{ cm.}$$

In the right triangle BHC,

$$CH^2 + HB^2 = CB^2 \qquad \text{(Pythagorean Theorem)}$$

$$CH^2 = 13^2 - 5^2$$

$$= 169 - 25$$

$$= 144$$

$$CH = 12 \text{ cm.}$$

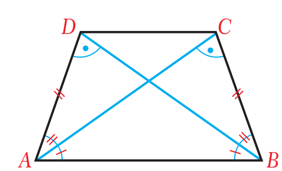
So the height of the trapezoid is 12 cm.

Theorem 26

The diagonals of an isosceles trapezoid are congruent.

Proof In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$ and $AD = BC$.

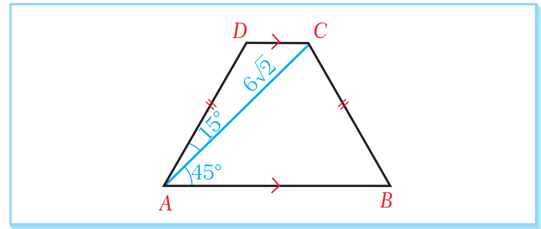
We have to prove $AC \cong BD$.



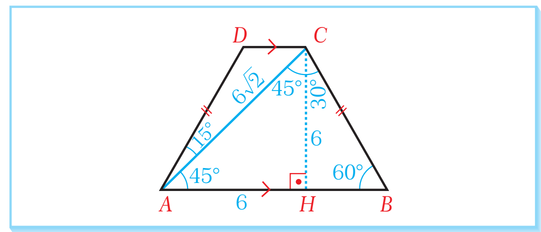
Statements	Reasons
1. $\angle A \cong \angle B$	Base angles of an isosceles trapezoid
2. $AD = BC$	Legs of an isosceles trapezoid
3. $AB = AB$	Common side of ΔABC and ΔBAD
4. $\Delta ABC \cong \Delta BAD$	SAS congruence postulate
5. $AC \cong BD$	Corresponding sides of congruent triangles are congruent.

We can also conclude that if the diagonals of a trapezoid are congruent then this trapezoid is an isosceles trapezoid.

EXAMPLE 64 $ABCD$ is an isosceles trapezoid with $AB \parallel DC$. If $m(\angle BAC) = 45°$, $m(\angle CAD) = 15°$ and $AC = 6\sqrt{2}$ cm, find the lengths of the sides of the trapezoid.



Solution Let us draw the altitude $CH \perp AB$, as in the figure. Then ΔAHC is an isosceles right triangle, and $AH = HC$. Also,



$$CH^2 + AH^2 = AC^2 \text{ (Pythagorean Theorem)}$$

$$2 \cdot CH^2 = (6\sqrt{2})^2$$

$$2 \cdot CH^2 = 72$$

$CH^2 = 36$, $CH = 6$ cm. So $AH = 6$ cm.

We also know $m(\angle A) = m(\angle B) = 60°$, since these are base angles of an isosceles trapezoid.

So in ΔCHB,

$$\cot 60° = \frac{HB}{CH}; \quad HB = CH \cdot \cot 60°$$

$$HB = 6 \cdot \frac{\sqrt{3}}{3} = 2\sqrt{3} \text{ cm, and}$$

$$\cos 60° = \frac{HB}{CB}; \quad CB = \frac{HB}{\cos 60°}$$

$$CB = \frac{2\sqrt{3}}{0{,}5} = 4\sqrt{3} \text{ cm.}$$

Finally, $AB = AH + HB = (6 + 2\sqrt{3})$ cm, and

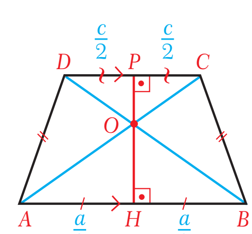
$$HB = \frac{AB - CD}{2}; \quad CD = AB - 2 \cdot HB$$

$$= 6 + 2\sqrt{3} - 4\sqrt{3}$$

$= (6 - 2\sqrt{3})$ cm. So the sides measure $(6 + 2\sqrt{3})$ cm and $(6 - 2\sqrt{3})$ cm.

Note

In an isoceles trapezoid, the perpendicular drawn from the midpoint of one base bisects the other base and passes through the intersection point of the diagonals. This line is called the **axis of symmetry** of the trapezoid. In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$, and PH is its axis of symmetry.

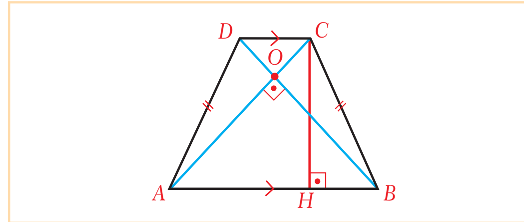


Theorem 27

If the diagonals of an isosceles trapezoid are perpendicular to each other then the height of the trapezoid is equal to half the sum of the lengths of the bases.

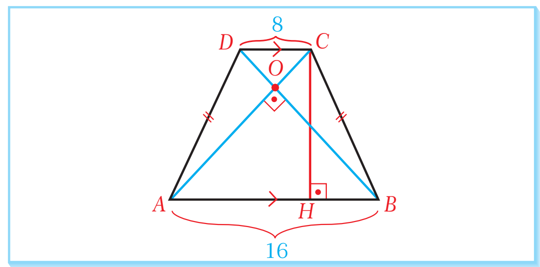
In the isosceles trapezoid $ABCD$ in the figure, $AB \parallel DC$, $AD = BC$ and $AC \perp DB$.

So by Theorem 27, $CH = \dfrac{AB + DC}{2}$.



EXAMPLE 65 An isosceles trapezoid has diagonals which are perpendicular to each other. Given that the bases measure 8 cm and 16 cm, find the height of this trapezoid.

Solution The figure shows an isosceles trapezoid $ABCD$ with $AB \parallel DC$.



Let us draw the altitude CH so $CH \perp AB$.

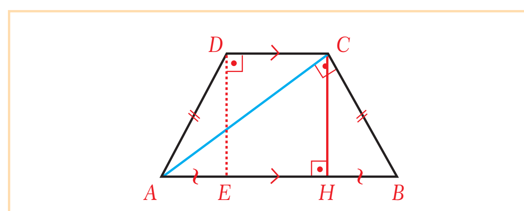
Since $AC \perp DB$, by Theorem 27 we can write

$$CH = \frac{AB + DC}{2}$$
$$= \frac{16+8}{2} = 12 \text{ cm}.$$

This is the height of the trapezoid.

Theorem 28

If the diagonals of an isosceles trapezoid are perpendicular to the legs then the height of the trapezoid is half the square root of the difference of the squares of the bases.

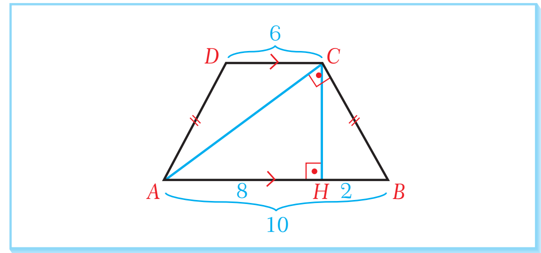


In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$ and $AC \perp BC$.

So by Theorem 28, $CH = \dfrac{\sqrt{AB^2 - DC^2}}{2}$.

EXAMPLE 66 $ABCD$ is an isosceles trapezoid with $AB \parallel DC$. Given $AC \perp BC$, $AB = 10$ cm and $DC = 6$ cm, find the height of this trapezoid.

Solution 1 Let us draw the altitude CH, so $CH \perp AB$. Since the diagonal AC is perpendicular to BC, by Theorem 28 we can write



$$CH = \frac{\sqrt{AB^2 - DC^2}}{2}$$

$$= \frac{\sqrt{10^2 - 6^2}}{2} = \frac{\sqrt{64}}{2} = 4 \text{ cm}.$$

Solution 2 $HB = \dfrac{AB - DC}{2} = \dfrac{10-6}{2} = 2$ cm

$AH = AB - HB$

$= 10 - 2 = 8$ cm

In the right triangle ACB,

$CH^2 = AH \cdot BH$ (first Euclidean theorem)

$= 8 \cdot 2$

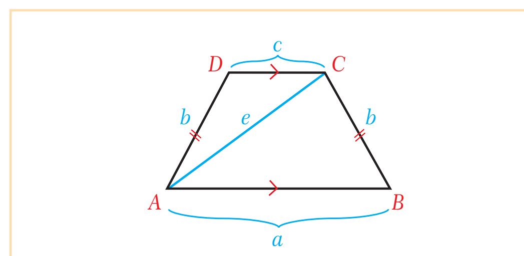
$= 16;\ CH = 4$ cm.

Theorem 29

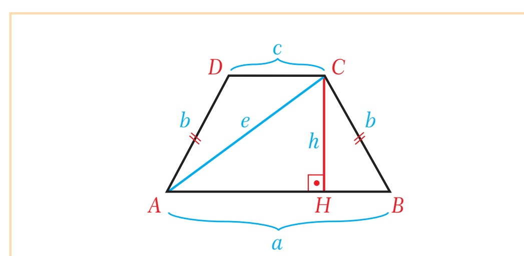
In an isosceles trapezoid, the difference of the squares of the lengths of a diagonal and a leg is equal to the product of the lengths of bases.

In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$, $AB = a$, $DC = c$, $BC = b$ and $AC = e$.

So by Theorem 29, $e^2 - b^2 = a \cdot c$.



Proof Let us draw the altitude $CH = h$, as shown in the figure.



By Theorem 25, $HB = \dfrac{a-c}{2}$.

Also, $AH = AB - HB$ so $AH = a - \dfrac{a-c}{2}$.

So $AH = \dfrac{a+c}{2}$.

Applying the Pythagorean Theorem to ΔAHC and ΔCHB gives us

$$e^2 = h^2 + \left(\frac{a+c}{2}\right)^2 \text{ and } b^2 = h^2 + \left(\frac{a-c}{2}\right)^2.$$

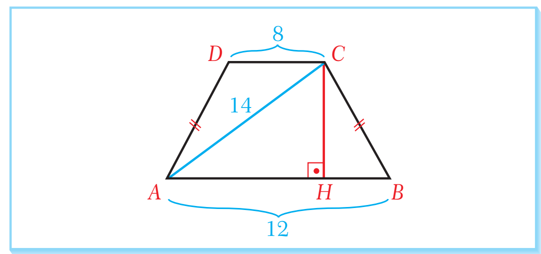
Subtracting the second equation from the first, we get

$$
\begin{aligned}
e^2 - b^2 &= \left(\frac{a+c}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2 \\
&= \left(\frac{a+c}{2} + \frac{a-c}{2}\right)\cdot\left(\frac{a+c}{2} - \frac{a-c}{2}\right) \\
&= \left(\frac{a+c+a-c}{2}\right)\cdot\left(\frac{a+c-a+c}{2}\right)
\end{aligned}
$$

$e^2 - b^2 = a \cdot c$, as required.

EXAMPLE 67 The bases of an isosceles trapezoid measure 8 cm and 12 cm respectively. If the diagonal of the trapezoid measures 14 cm, find the length of its legs.

Solution 1 By Theorem 29 we have



$$
\begin{aligned}
AC^2 - CB^2 &= AB \cdot CD \\
CB^2 &= 14^2 - 12 \cdot 8 \\
&= 196 - 96 \\
&= 100;\ \ CB = 10 \text{ cm.}
\end{aligned}
$$

Solution 2 Let us draw the altitude CH, so $CH \perp AB$. Then

$$HB = \frac{AB - DC}{2} = \frac{12-8}{2} = 2 \text{ cm.}$$

Also, $AH = AB - HB$

$= 12 - 2 = 10$ cm.

In the right triangle ACH,

$CH^2 = AC^2 - AH^2$ (Pythagorean Theorem)

$= 14^2 - 10^2$

$= 96;\ \ CH = \sqrt{96}$ cm.

In the right triangle BCH,

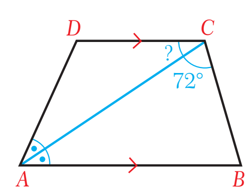
$BC^2 = CH^2 + BH^2$ (Pythagorean Theorem)

$= 96 + 2^2$

$= 100;\ \ BC = 10$ cm. This is the length of the leg.

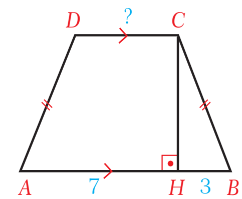
Expensive trapezoids

Check Yourself 16



1. In the figure, $ABCD$ is an isosceles trapezoid with $DC \parallel AB$, and the diagonal AC bisects $\angle A$. If $m(\angle BCA) = 72°$, find $m(\angle DCA)$.

2. In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$, $CH \perp AB$, $AH = 7$ cm and $HB = 3$ cm. Find the length of DC.



3. $ABCD$ is an isosceles trapezoid with $AB \parallel DC$, $AB = 20$ cm, $BC = 10$ cm and $CD = 8$ cm. Find the length of BD.

Answers

1. 36°   2. 4 cm   3. $2\sqrt{65}$ cm

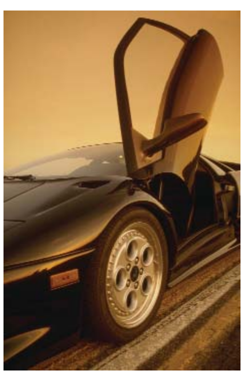
4. Right Trapezoids

a. Definition

Definition

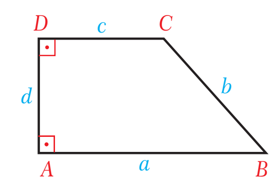
right trapezoid

A **right trapezoid** is trapezoid which contains a right angle.



In the figure, $AB \parallel DC$ and $m(\angle A) = m(\angle D) = 90°$.

So by the definition, $ABCD$ is a right trapezoid.



b. Properties of a right trapezoid

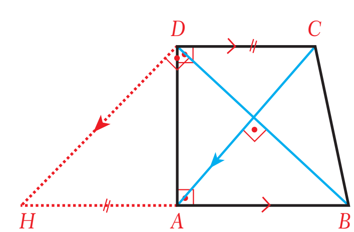
A right trapezoid has all the properties of an ordinary trapezoid, and also some additional properties. Let us look at one important property.

Theorem 30

If the diagonals of a right trapezoid are perpendicular to each other then the height is equal to the square root of the product of the lengths of the bases.

Proof In the figure, $ABCD$ is a right trapezoid with $AB \parallel DC$, $AD \perp AB$ and $AC \perp DB$.

We need to prove that $AD = \sqrt{AB \cdot DC}$.



Let us draw DH from point D such that $AC \parallel DH$ and H is a point on the extension of AB.

If one of two parallel lines is perpendicular to a line ℓ then the other parallel line is also perpendicular to ℓ.

Then, since $DH \parallel AC$ and $AC \perp DB$ we have $DH \perp DB$.

Also, $DC = HA$ since $ACDH$ is a parallelogram.

In the right triangle HDB,

$$DA^2 = HA \cdot AB \qquad \text{(second Euclidean theorem)}$$

$$DA = \sqrt{HA \cdot AB} = \sqrt{DC \cdot AB}. \qquad (HA = DC)$$

So $DA = \sqrt{DC \cdot AB}$ as required.

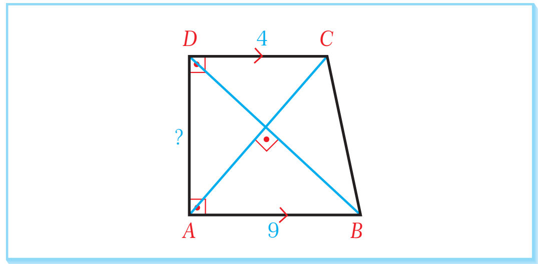
EXAMPLE 68 A right trapezoid has perpendicular diagonals and base lengths 4 cm and 9 cm. Find the height of this trapezoid.

Solution In the figure, $ABCD$ is a right trapezoid with $AB \parallel DC$, $AD \perp AB$ and $AC \perp DB$.

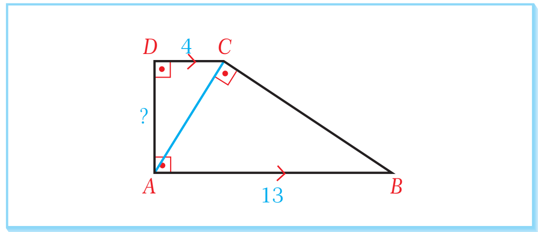
By Theorem 30, $DA = \sqrt{DC \cdot AB}$

$= \sqrt{4 \cdot 9}$

$= 6$ cm.



EXAMPLE 69 In the figure, $ABCD$ is a right trapezoid with $AB \parallel DC$ and $AC \perp BC$. Given $AB = 13$ cm and $DC = 4$ cm, find the height of the trapezoid.

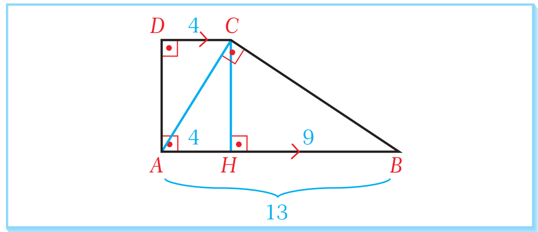


Solution Let us draw the altitude CH. Then

$AH = DC = 4$ cm (opposite sides of a rectangle)

$HB = AB - AH$

$= 13 - 4 = 9$ cm.



In the right triangle BCA,

$CH^2 = HA \cdot HB$ (first Euclidean theorem)

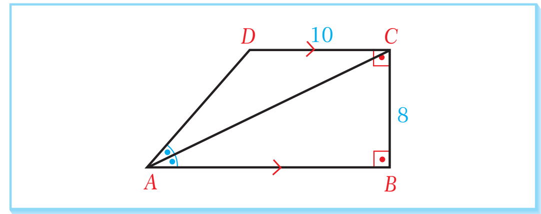
$= 4 \cdot 9$

$= 36$

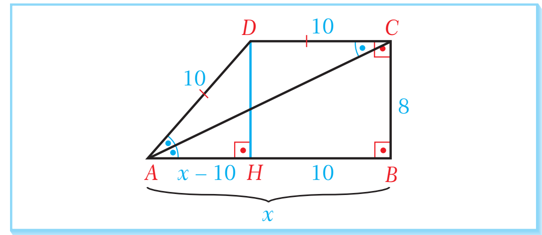
$CH = 6$ cm. This is the height of the trapezoid.

EXAMPLE 70 In the figure, $ABCD$ is a right trapezoid with $AB \parallel DC$. AC is the bisector of $\angle A$, $BC = 8$ cm and $DC = 10$ cm.

Find the length of AB.



Solution $m(\angle BAC) = m(\angle ACD)$ because they are alternate interior angles. So ΔADC is isosceles and $AD = DC = 10$ cm.



Let us draw the altitude DH. Then

$DH = BC = 8$ cm (opposite sides of rectangle $HBCD$)

$HB = DC = 10$ cm (opposite sides of rectangle $HBCD$)

$AH = AB - HB$.

Let $AB = x$ with $x > 10$. So $AH = x - 10$.

In the right triangle AHD,

$AD^2 = DH^2 + AH^2$ (Pythagorean Theorem)

$10^2 = (x - 10)^2 + 8^2$

$100 = x^2 - 20x + 100 + 64$

$x^2 - 20x + 64 = 0$

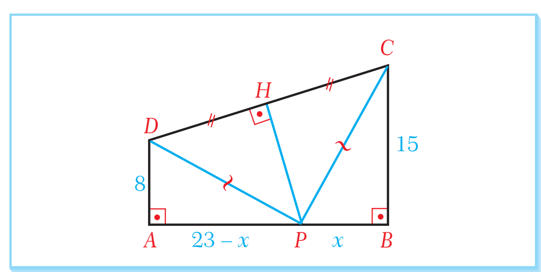
$(x - 4)(x - 16) = 0$ (factorize)

$x = 4$ or $x = 16$.

$x = 4$ cannot be a solution because $x > 10$. So $AB = x = 16$ cm.

EXAMPLE 71 In a right trapezoid $ABCD$, $AD \parallel BC$ and $AD \perp AB$. $AD = 8$ cm, $BC = 15$ cm and $AB = 23$ cm are given. Additionally, point H is the midpoint of side DC and point P is on the side AB such that $PH \perp DC$. Find the length of PB.

Solution Remember that in an isosceles triangle, the median to the base is also the altitude to the base.



Let us draw DP and PC as in the figure. Then ΔDPC is an isosceles triangle, because $PH \perp DC$ and $DH = HC$.

So $DP = PC$.

Also, $AP = AB - PB$.

Let $PB = x$, then $AP = 23 - x$.

In the right triangles PAD and PBC,

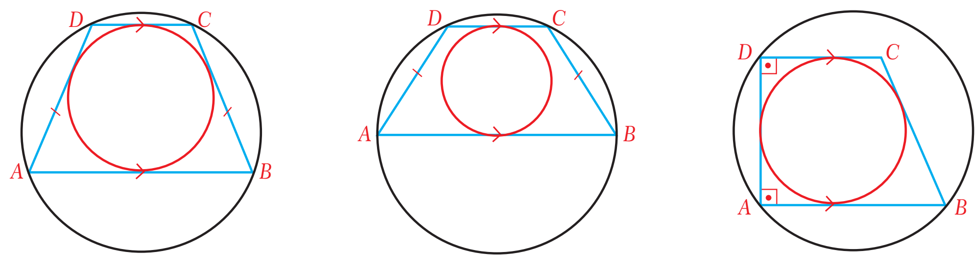
$$PD^2 = AD^2 + AP^2 \text{ and } PC^2 = PB^2 + BC^2 \qquad \text{(Pythagorean Theorem)}$$
$$AD^2 + AP^2 = PB^2 + BC^2 \qquad (PD = PC)$$
$$8^2 + (23 - x)^2 = x^2 + 15^2$$
$$64 + 529 - 46x + \cancel{x^2} = \cancel{x^2} + 225$$
$$46x = 593 - 225$$
$$46x = 368$$
$$x = 8 \text{ cm.}$$

So $PB = 8$ cm.

Note

A trapezoid can be both an inscribed and circumscribed quadrilateral. An isosceles trapezoid is always an inscribed quadrilateral but not always a circumscribed quadrilateral. A right trapezoid is never an inscribed quadrilateral, but it may be a circumscribed quadrilateral.

In each figure, $ABCD$ is a trapezoid.



inscribed and circumscribed

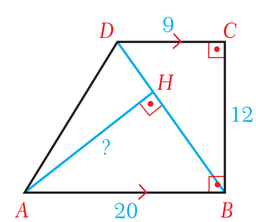
inscribed but not circumscribed

circumscribed but not inscribed

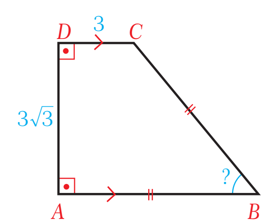
Check Yourself 17

1. The bases of a right trapezoid measure 5 cm and 8 cm respectively. If the height of the trapezoid is 4 cm, find the lengths of its legs.

2. In the figure, $ABCD$ is a right trapezoid, H is a point on the diagonal DB and $AH \perp DB$. $AB = 20$ cm, $BC = 12$ cm and $DC = 9$ cm are given. Find the length of AH.



3. In the figure, $ABCD$ is a right trapezoid with $AB = BC$, $CD = 3$ cm and $AD = 3\sqrt{3}$ cm. Find $m(\angle ABC)$.



Answers

1. 4 cm and 5 cm 2. 16 cm 3. 60°

G. KITE

1. Definition

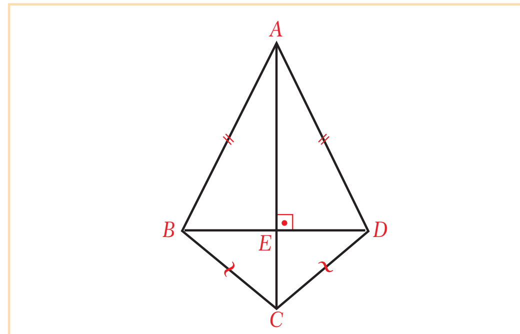
Definition

kite

A **kite** is a quadrilateral with two pairs of consecutive congruent sides and two non-congruent opposite sides.

Not all flying kites are in the shape of a kite!

In the figure, $AB \cong AD$ and $BC \cong CD$, and also $AB \neq CD$. So the quadrilateral $ABCD$ is a kite.



As we can see, a kite consists of two isosceles triangles with a common base BD.

A square and a rhombus can also be divided into two isosceles triangles with a common base. Therefore, the properties of a kite are similar to some of the properties of a rhombus and a square.

2. Properties of a Kite

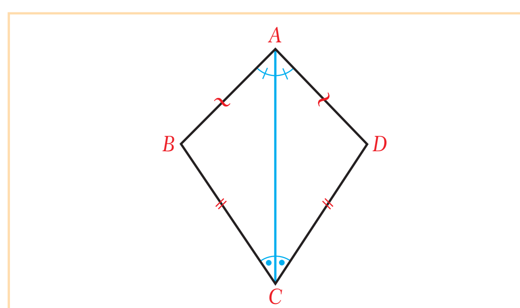
Theorem 31

The two angles formed by the non-congruent sides of a kite are congruent angles.

Proof

SSS (Side Side Side) postulate: If the three sides of one triangle can be paired with the three sides of another triangle such that the sides in each pair are congruent, then the triangles are congruent.

In the figure, $ABCD$ is a kite, $AB = AD$ and $CB = CD$.



We need to show that $\angle B \cong \angle D$.

Let us draw the diagonal AC. Since

$AB = AD$, (given)

$CB = CD$ (given)

and $AC = AC$,

then by the SSS postulate, $\Delta ABC \cong \Delta ADC$.

So $\angle B \cong \angle D$, as required.

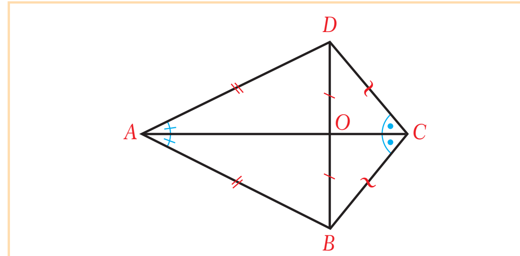
Theorem 32

The diagonals of a kite are perpendicular.

Proof In the figure, $ABCD$ is a kite, $AB = AD$ and $CB = CD$.

We need to show that $AC \perp BD$.

Let us draw the diagonals AC and BD. Then $\Delta ABC \cong \Delta ADC$ by the SSS postulate. Therefore,



$\angle DCA \cong \angle BCA$ (corresponding angles of congruent triangles)

$\angle CDB \cong \angle CBD$ (base angles of the isosceles triangle DCB)

$$m(\angle CDB) + m(\angle CBD) + m(\angle BCD) = 180° \quad \text{(sum of interior angles)}$$

$$2 \cdot m(\angle CDB) + 2 \cdot m(\angle OCD) = 180°$$

$$m(\angle CDB) + m(\angle OCD) = 90°.$$

In ΔDOC,

$$m(\angle CDB) + m(\angle OCD) + m(\angle DOC) = 180° \quad \text{(sum of interior angles)}$$

$$90° + m(\angle DOC) = 180°$$

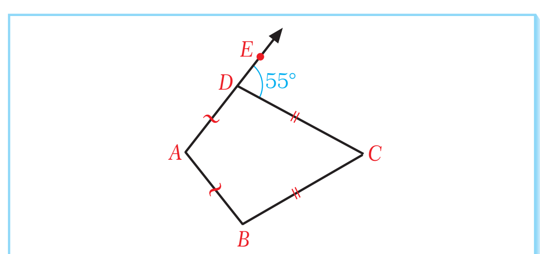
$$m(\angle DOC) = 90°.$$

So $AC \perp BD$, which is the required result.

Notice also that since ΔDCB is isosceles and $DB \perp AC$, $DO = OB$.

EXAMPLE 72 In the figure opposite, $ABCD$ is a kite with $AB = AD$ and $CB = CD$.

If $m(\angle A) = 4 \cdot m(\angle C) - 10°$ and $m(\angle EDC) = 55°$, find the measure of each interior angle of the kite.



Solution $m(\angle ADC) + m(\angle EDC) = 180°$ (supplementary angles)

$$m(\angle ADC) = 180° - 55°$$
$$= 125°$$

$m(\angle ADC) = m(\angle B) = 125°$

In quadrilateral $ABCD$,

$$m(\angle ADC) + m(\angle B) + m(\angle A) + m(\angle C) = 360° \quad \text{(sum of interior angles)}$$
$$2 \cdot 125° + 4 \cdot m(\angle C) - 10° + m(\angle C) = 360°$$
$$5 \cdot m(\angle C) = 120°$$
$$m(\angle C) = 24°.$$

Finally, $m(\angle A) = 4 \cdot m(\angle C) - 10°$ (given)

$$= 4 \cdot 24° - 10°$$
$$= 86°.$$

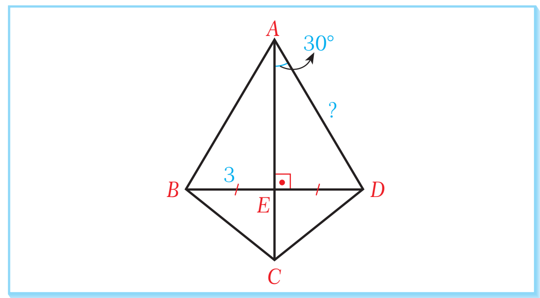
EXAMPLE 73 In a kite $ABCD$, $AB = AD$, $CB = CD$ and E is the intersection point of the diagonals. If $m(\angle EAD) = 30°$ and $BE = 3$ cm, find the length of AD.

Solution In the figure, $BD \perp AC$ and $BE = ED$.

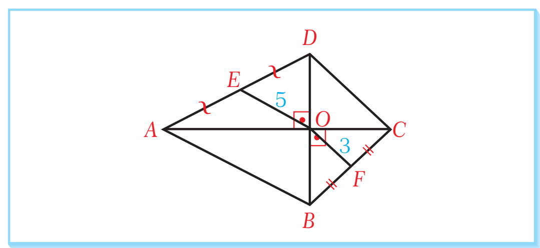
So $ED = 3$ cm.

In the right triangle AED, side ED is opposite the 30° angle and we know from trigonometry that the length of the side opposite 30° is half the length of hypotenuse.

So $AD = 2ED = 6$ cm.



EXAMPLE 74 In the figure, $ABCD$ is a kite, $AB = AD$ and $CB = CD$. Points E and F are the midpoints of sides AD and CB respectively. Find the perimeter of the kite if $EO = 5$ cm and $OF = 3$ cm.



Solution The diagonals of a kite are perpendicular to each other, so $BD \perp AC$.

Also, OE and OF are medians of the right triangles AOD and COB. In a right triangle, the length of the median to the hypotenuse is equal to half the length of the hypotenuse.

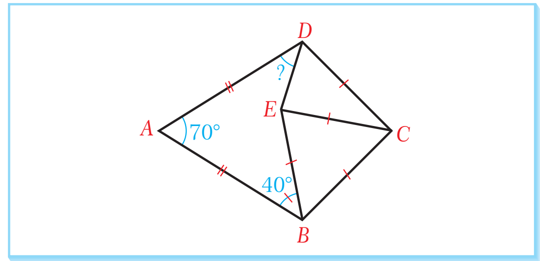
So $AE = ED = EO = 5$ cm, and $AD = 10$ cm.

Similarly, $BF = FC = OF = 3$ cm and so $BC = 6$ cm.

So $P(ABCD) = AB + AD + CB + CD$

$$= 2AB + 2BC \qquad (AB = AD \text{ and } BC = CD)$$
$$= 2(10 + 6)$$
$$= 32 \text{ cm}.$$

EXAMPLE 75 In the figure, $ABCD$ is a kite, $AB = AD$, $CB = CD$ and ΔBEC is an equilateral triangle. If $m(\angle A) = 70°$ and $m(\angle ABE) = 40°$, find $m(\angle ADE)$.



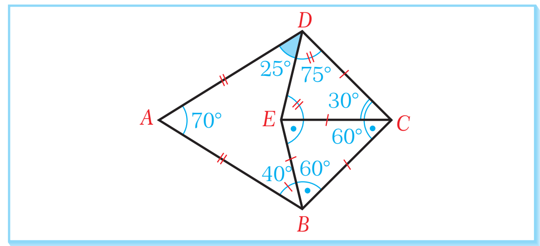
Solution $DC = BC = EC = EB$ (sides of an equilateral triangle)

$m(\angle EBC) = m(\angle BCE) = 60°$ (interior angles of an equilateral triangle)

$m(\angle ABC) = m(\angle ABE) + m(\angle CBE)$

$= 40° + 60°$

$= 100°$



$m(\angle B) = m(\angle D) = 100°$

In quadrilateral $ABCD$,

$$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360° \quad \text{(sum of interior angles)}$$

$$2 \cdot 100° + m(\angle C) + 70° = 360°$$

$$m(\angle C) = 90°$$

$$m(\angle DCE) = m(\angle DCB) - m(\angle ECB)$$

$$= 90° - 60°$$

$$= 30°.$$

ΔDCE is isosceles because $EC = CD$, and so

$$m(\angle DEC) + m(\angle EDC) + m(\angle ECD) = 180° \quad \text{(sum of interior angles)}$$

$$2 \cdot m(\angle EDC) + 30° = 180°$$

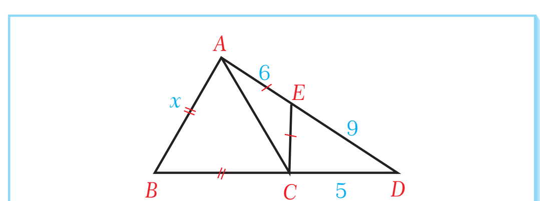
$$m(\angle EDC) = 75°.$$

Finally, $m(\angle ADE) = m(\angle ADC) - m(\angle EDC)$

$= 100° - 75°$

$= 25°.$

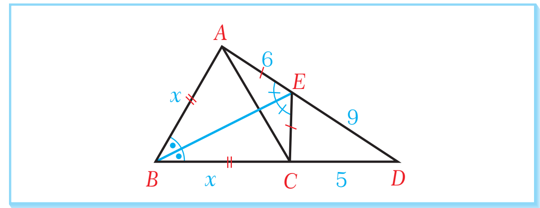
EXAMPLE 76 In the figure, $ABCE$ is a kite with $AE = EC$. Given $AE = 6$ cm, $ED = 9$ cm, $CD = 5$ cm and $AB = x$, find the value of x.



Solution We know $AB = BC = x$ since $ABCE$ is a kite.

Let us draw the diagonal BE.

Then BE bisects $\angle ABC$ and $\angle CEA$.



So by the angle bisector theorem in ΔABD we have

$$\frac{AB}{BD} = \frac{AE}{ED}$$

$$\frac{x}{x+5} = \frac{6}{9}$$

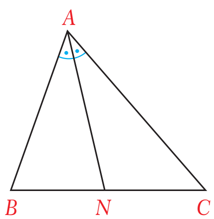
$$9x = 6x + 30$$

$$3x = 30$$

$$x = 10 \text{ cm.}$$

Angle bisector theorem:

If a line segment bisects an angle of a triangle then it divides the opposite side into segments proportional to the other two sides of the triangle.

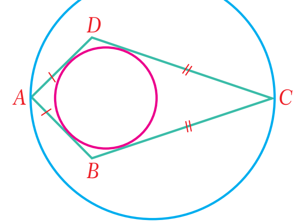


$$\frac{BN}{NC} = \frac{AB}{AC}$$

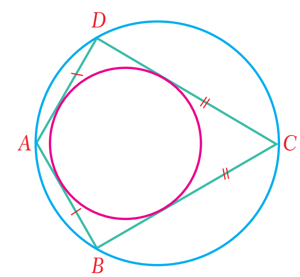
The pattern of the Piazza del Campidoglio in Rome includes many kite shapes.

Note

A kite is always a circumscribed quadrilateral, but it is not always an inscribed quadrilateral because opposite angles of a kite may not be supplementary. In the figures, $ABCD$ is a kite.



circumscribed but not inscribed

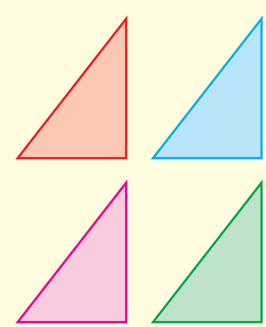


inscribed and circumscribed

Activity

Cut four congruent right triangles from a piece of paper. Show how the four triangles can be put together to make each quadrilateral.

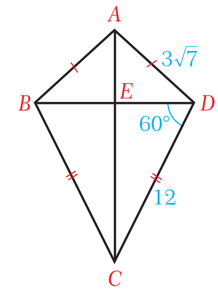
1. a rhombus
2. a rectangle
3. a parallelogram that is neither a rhombus nor a rectangle
4. a trapezoid
5. a kite



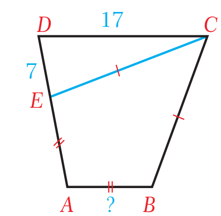
Check Yourself 18

1. In a kite $ABCD$, $AB = AD$ and $CB = CD$. If $m(\angle B) = 40°$ and $m(\angle C) = 110°$, find $m(\angle D)$.

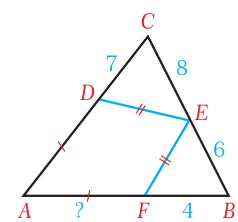
2. In the figure, $ABCD$ is a kite, $AB = AD$ and $m(\angle BDC) = 60°$. If $CD = 12$ cm and $DA = 3\sqrt{7}$ cm, find the length of AC.



3. In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$, and $ABCE$ is a kite with $EA = AB$. If $CD = 17$ cm and $DE = 7$ cm, find the length of AB.

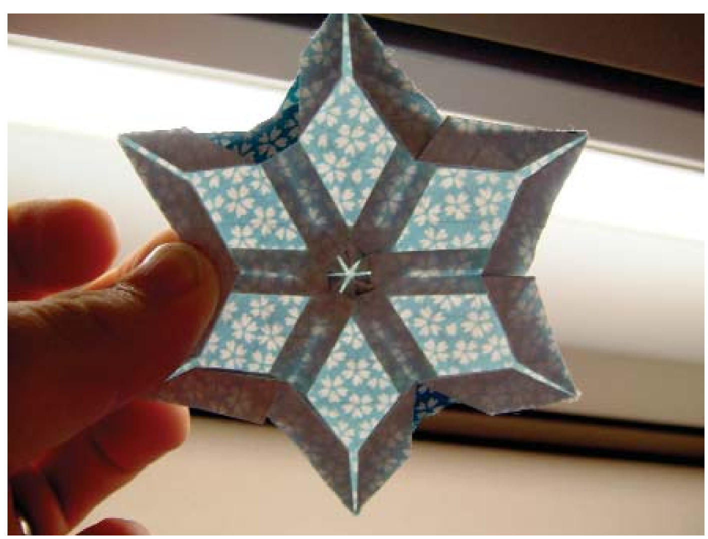


4. In the figure, ABC is a triangle and $AFED$ is a kite with $FE = ED$. Points F, E and D are on the sides AB, BC and CA respectively, and $FB = 4$ cm, $BE = 6$ cm, $EC = 8$ cm and $CD = 7$ cm are given. Find the length of AF.

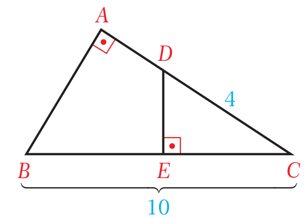


Answers

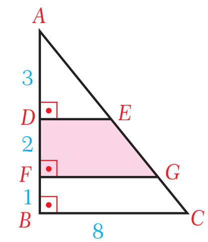
1. 40° 2. $9\sqrt{3}$ cm 3. 10 cm 4. 5 cm



3. In the figure,
$BC = 10$ cm and $DC = 4$ cm.
Find the value of the ratio $\dfrac{A(\Delta DEC)}{A(\Delta ABC)}$.



4. What is the area of quadrilateral $FGED$ in the figure?



5. Prove Property 9.3: if $\Delta ABC \sim \Delta DEF$ then $\dfrac{A(\Delta ABC)}{A(\Delta DEF)} = k^2$.

Answers

1. 9 2. 18 cm 3. $\dfrac{4}{25}$ 4. $\dfrac{32}{3}$

H. THE TRIANGLE PROPORTIONALITY THEOREM AND THALES' THEOREM

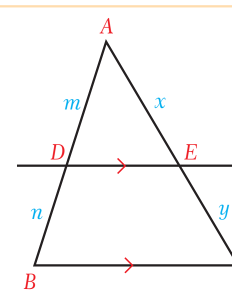
1. The Triangle Proportionality Theorem

Theorem

Triangle Proportionality Theorem

A line parallel to one side of a triangle which intersects the other two sides of the triangle at different points divides these two sides proportionally. In other words, in the figure below, $\dfrac{m}{n} = \dfrac{x}{y}$.

Proof



Look at the figure.

Given: $DE \parallel BC$

Prove: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

$\Delta ABC \sim \Delta ADE$ (AA Similarity Postulate)

So $\dfrac{AB}{AD} = \dfrac{AC}{AE}$.

Let $AD = m$, $DB = n$, $AE = x$ and $EC = y$.

Then $\dfrac{m+n}{m} = \dfrac{x+y}{x}$

$$1+\frac{n}{m} = 1+\frac{y}{x}$$

$\dfrac{n}{m} = \dfrac{y}{x}$. So $\dfrac{DB}{AD} = \dfrac{EC}{AE}$, and so $\dfrac{AD}{DB} = \dfrac{AE}{EC}$, as required.

Conclusion

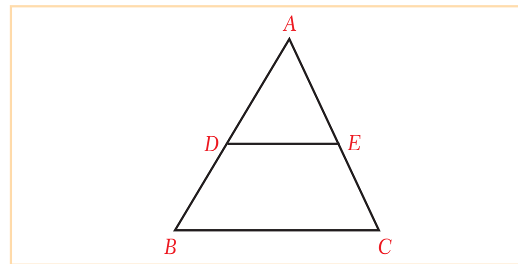
Using the properties of ratio in the previous figure, we can conclude that if DE is parallel to BC then $\frac{AD}{DB} = \frac{AE}{EC}$, $\frac{AB}{DB} = \frac{AC}{EC}$ and $\frac{AB}{AD} = \frac{AC}{AE}$.

Theorem

Converse of the Triangle Proportionality Theorem

If a line divides two sides of a triangle proportionally then it is parallel to the third side of the triangle.

Proof Look at the figure.

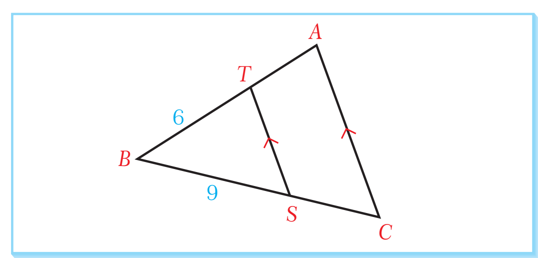


Given: $\frac{AD}{DB} = \frac{AE}{EC}$

Prove: $DE \parallel BC$

- $\frac{AD}{DB} = \frac{AE}{EC}$ — Given
- $\frac{DB}{AD} + \frac{AD}{AD} = \frac{EC}{AE} + \frac{AE}{AE}$ — Properties of proportion
- $\frac{AB}{AD} = \frac{AC}{AE}$ — Simplification (using the figure)
- $\angle BAC \cong \angle DAE$ — Common angle
- $\Delta ABC \sim \Delta ADE$ — SAS Similarity Theorem
- $\angle D \cong \angle B$, $\angle E \cong \angle C$ — Definition of similarity
- $DE \parallel BC$ — Converse of the Corresponding Angles Theorem

EXAMPLE 77 In the figure, $TS \parallel AC$, $BT = 6$ cm, $BS = 9$ cm, $AB = 2x + 4$ and $BC = 5x$. Find SC.



Solution Since $TS \parallel AC$, by the Triangle Proportionality Theorem we can write

$$\frac{BT}{BA} = \frac{BS}{BC}$$

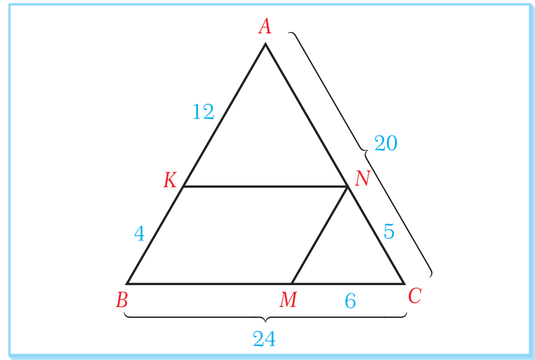
$$\frac{6}{2x+4} = \frac{9}{5x}$$

$$10x = 6x + 12;\ \ 4x = 12;\ \ x = 3 \text{ cm.}$$

So $BC = 5x = 5 \cdot 3 = 15$ cm and

$SC = BC - BS = 15 - 9 = 6$ cm.

EXAMPLE 78 In the figure, $AK = 12$ cm, $KB = 4$ cm, $AC = 20$ cm, $NC = 5$ cm, $BC = 24$ cm and $MC = 6$ cm. Show that $KN \parallel BC$ and $MN \parallel AB$.



Solution To show that the lines are parallel, it is sufficient by the Converse of the Triangle Proportionality Theroem to show that

$\frac{AK}{KB} = \frac{AN}{NC}$ and $\frac{BM}{MC} = \frac{AN}{NC}$.

Since $AN = AC - NC$,

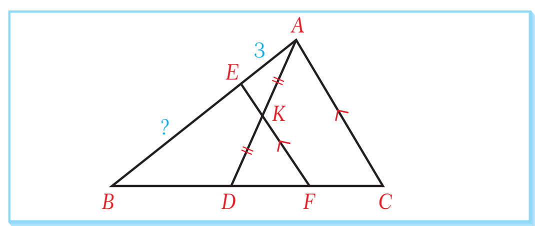
$AN = 20 - 5 = 15$ cm.

Similarly, $BM = 18$ cm.

So $\frac{AK}{KB} = \frac{12}{4} = 3$ and $\frac{AN}{NC} = \frac{15}{5} = 3$. So $\frac{AK}{KB} = \frac{AN}{NC}$, and so by the Converse of the Triangle Proportionality Theorem, $KN \parallel BC$.

Also, $\frac{BM}{MC} = \frac{18}{6} = 3$ and $\frac{AN}{NC} = \frac{15}{5} = 3$, so $\frac{BM}{MC} = \frac{AN}{NC}$ and so by the same theorem, $MN \parallel AB$.

EXAMPLE 79 In the triangle ABC at the right, $EF \parallel AC$, $AK = KD$, $BD = 2DC$ and $AE = 3$ cm. Find the length of segment EB.



Solution In ΔADC, $\frac{DK}{AK} = \frac{DF}{FC} = 1$. (Triangle Proportionality Theorem and $AK = KD$)

So $DF = FC$.

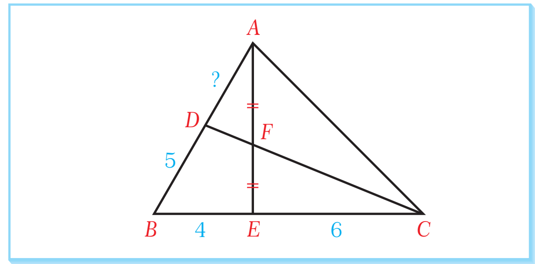
Let $DF = FC = y$, then $BD = 2CD = 4y$.

So in ΔABC we have $\frac{AE}{EB} = \frac{FC}{BF}$ (Triangle Proportionality Theorem)

$$\frac{3}{EB} = \frac{y}{5y}$$

$$EB = 15 \text{ cm.}$$

EXAMPLE 80 In ΔABC at the right, $AF = FE$, $DB = 5$ cm, $BE = 4$ cm and $EC = 6$ cm. Find the length of AD.



Solution First we find a point K on AB such that $DC \parallel KE$. Then in ΔDBC,

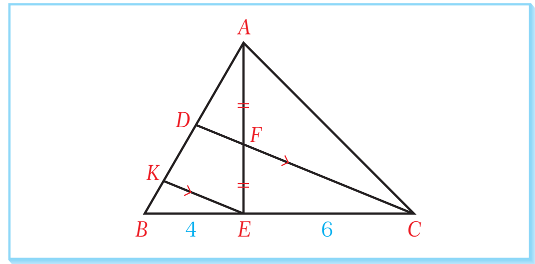
$$\frac{BE}{EC} = \frac{BK}{KD}. \text{ (Triangle Proportionality Theorem)}$$

$$\text{So } \frac{4}{6} = \frac{BK}{5 - BK}$$

$$6 \cdot BK = 20 - 4 \cdot BK$$

$$10 \cdot BK = 20$$

$$BK = 2 \text{ cm.}$$



Hence $KD = BD - BK = 5 - 2 = 3$ cm.

On the other hand, in ΔAKE we have

$$\frac{AF}{FE} = \frac{AD}{DK} = 1. \text{ (Triangle Proportionality Theorem and } AF = FE)$$

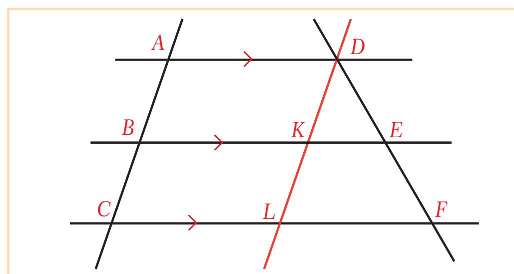
So $AD = 3$ cm.

2. Thales' Theorem of Parallel Lines

Theorem

Thales' Theorem

If two transversals intersect three or more parallel lines then the parallel lines divide the transversals proportionally. This theorem is known as **Thales' Theorem**.



Proof Look at the figure.

Given: $AD \parallel BE \parallel CF$

Prove: $\frac{AB}{BC} = \frac{DE}{EF}$

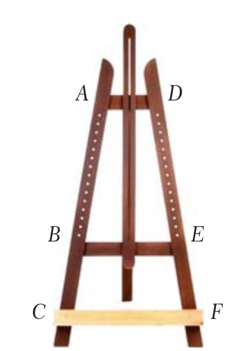
First we draw a line which is parallel to AC and passes through D. Let us label the intersection points K and L of this new line with BE and CF.

Then $BKDA$ and $CLKB$ are parallelograms, since if the opposite sides of a quadrilateral are parallel then the quadrilateral is a parallelogram. So $DK = AB$ and $KL = BC$. (1)

Since $KE \parallel LF$, by the Triangle Proportionality Theorem in ΔDLF we have $\frac{DK}{KL} = \frac{DE}{EF}$. (2)

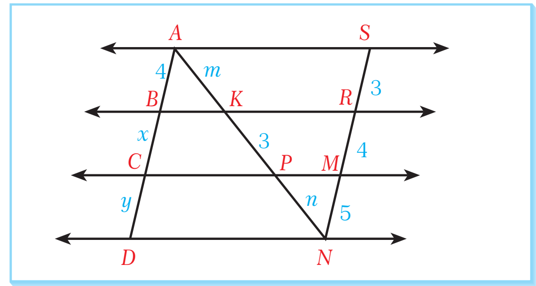
Substituting (1) into (2) gives us $\frac{AB}{BC} = \frac{DE}{EF}$, as required.

Can you see the proportional lengths?



EXAMPLE 81 In the figure, $AS \parallel BR \parallel CM \parallel DN$.

Find the lengths m, n, x and y using the information in the figure.



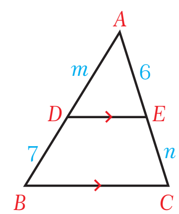
Solution Since $AS \parallel BR \parallel CM \parallel DN$ and AD, AN and NS are tranversals, we can apply Thales' Theorem:

$$\frac{AK}{KP} = \frac{SR}{RM};\ \frac{m}{3} = \frac{3}{4};\ m = \frac{9}{4} \text{ and } \frac{KP}{PN} = \frac{RM}{MN};\ \frac{3}{n} = \frac{4}{5};\ n = \frac{15}{4}.$$

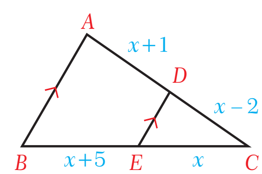
$$\frac{AB}{BC} = \frac{SR}{RM};\ \frac{4}{x} = \frac{3}{4};\ x = \frac{16}{3} \text{ and } \frac{AB}{CD} = \frac{SR}{MN};\ \frac{4}{y} = \frac{3}{5};\ y = \frac{20}{3}.$$

Check Yourself 7

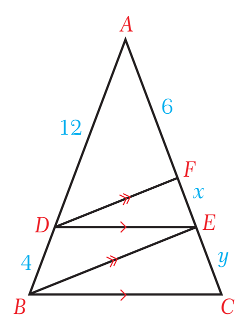
1. Find the value of $m \cdot n$ in the figure.



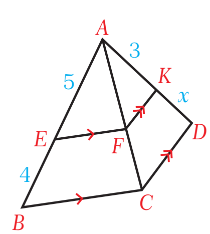
2. In the figure, $DE \parallel AB$.
   Find the value of x.



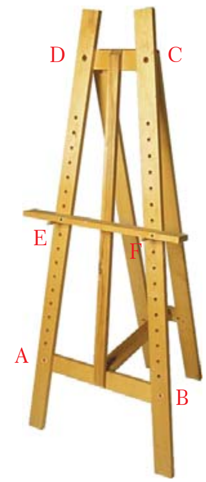
3. In the figure, $DE \parallel BC$, $DF \parallel BE$, $AD = 12$, $DB = 4$ and $AF = 6$. Find the lengths x and y.



4. Find the value of x in the figure.



5. In the figure, $DC \parallel EF \parallel AB$. $DE = 50$ cm, $EA = 70$ cm, $CF = x$ and and $FB = x + 20$ cm are given. Find the value of x.



Answers

1. 42 2. 5 3. $x = 2,\ y = \dfrac{8}{3}$ 4. $\dfrac{12}{5}$ 5. 50 cm

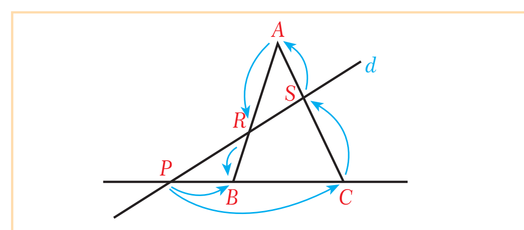
I. FURTHER APPLICATIONS

1. Menelaus' Theorem

Theorem

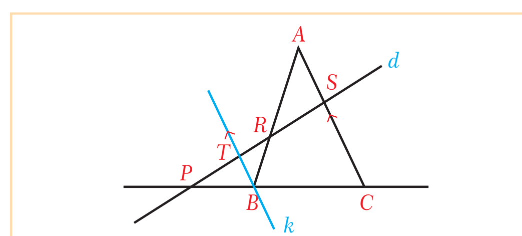
Menelaus' Theorem

Let ABC be a triangle. If a line d intersects the two sides AB and AC and the extension of the third side BC of ΔABC at points R, S and P respectively, then $\dfrac{PB}{PC} \cdot \dfrac{CS}{SA} \cdot \dfrac{AR}{RB} = 1$.



Proof

Let us draw the line k through point B and parallel to side AC (Parallel Postulate), and let T be the intersection point of lines k and d.



Then $\Delta PBT \sim \Delta PCS$ by the AA Similarity Postulate.

So $\dfrac{PB}{PC} = \dfrac{BT}{CS}$. (1)

Moreover, $\Delta BRT \sim \Delta ARS$ by the AA Similarity Postulate.

So $\dfrac{BR}{AR} = \dfrac{BT}{AS}$. (2)

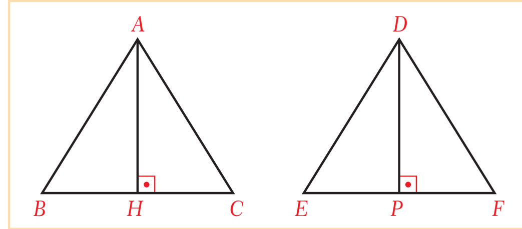
Dividing (1) by (2) side by side gives $\dfrac{\frac{PB}{PC}}{\frac{BR}{AR}} = \dfrac{\frac{BT}{CS}}{\frac{BT}{AS}}$; $\dfrac{PB}{PC} \cdot \dfrac{AR}{BR} = \dfrac{AS}{CS}$; $\dfrac{PB}{PC} \cdot \dfrac{AR}{BR} \cdot \dfrac{CS}{AS} = 1$.

So $\dfrac{PB}{PC} \cdot \dfrac{CS}{SA} \cdot \dfrac{AR}{RB} = 1$, as required.

Menelaus of Alexandria (c. 40-140 AD) was a Greek mathematician and astronomer. He was the first mathematician to describe a spherical triangle, and proved the theorem described here in his book *Sphaerica*, which is the only book he wrote that has survived.

3. Look at the figure.

Given: $\Delta ABC \cong \Delta DEF$, and AH and DP are the altitudes to sides BC and EF respectively.



Prove: $AH = DP$

$\angle BAH \cong \angle EDP$	($\angle B \cong \angle E$ and $\angle BHA \cong \angle EPD = 90°$)
$AB \cong DE$	(CPCTC)
$\angle B \cong \angle E$	(CPCTC)
$\Delta ABC \cong \Delta DEF$	(ASA Congruence Theorem)

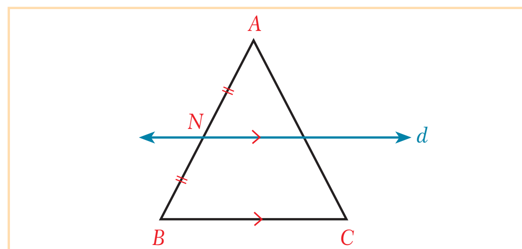
So $AH = DP$ because CPCTC.

Theorem

If a line parallel to one side of a triangle bisects another side of the triangle, it also bisects the third side.

Proof

Let us draw an appropriate figure.



Given: Line d bisects AB and $d \parallel BC$.

Prove: Line d bisects AC.

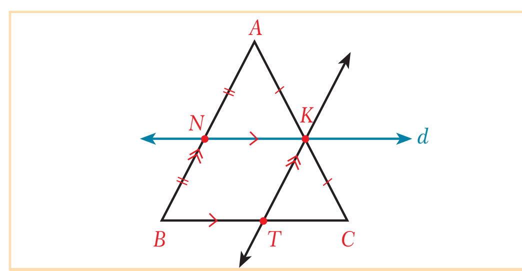
Pasch's Postulate

If a line intersects one side of a triangle, then it must also intersect one of the other two sides.

- Since line d bisects AB and is parallel to BC, by Pasch's Postulate it cuts side AC of the triangle. Let K be the point of intersection with AC.

  Now we have to show that $AK = KC$.

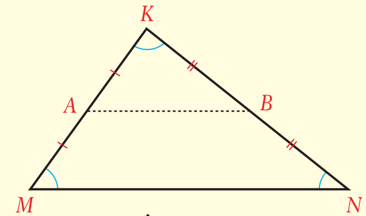
Let us draw a line through K which is parallel to AB and cuts BC at the point T, as shown in the figure below.



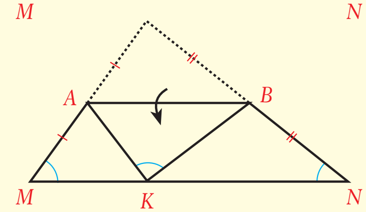
- Since parallel line segments between two parallel lines are congruent, $KT = NB$.
- $KT = AN$ because $AN = NB$.
- $\angle KTC \cong \angle ABC$ because they are corresponding angles.
- $\angle CKT \cong \angle KAN$ because they are also corresponding angles.
- So by the ASA Congruence Theorem, $\Delta KCT \cong \Delta AKN$.
- $AK \cong KC$ because CPCTC. So $AK = KC$.

Activity — Paper Folding - Midsegments in a Triangle

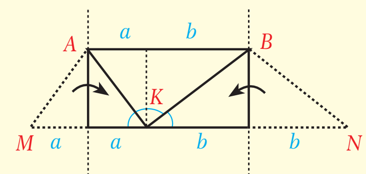
- Cut out a triangle and label its vertices K, M and N.



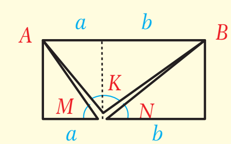
- Fold M onto K to find the midpoint A of KM. Similarly, fold N to K to find the midpoint B of KN.



- Fold K to MN on AB. Then fold M to K and fold N to K.



What can you say about the lines AB and MN in relation to each other? How do their lengths compare? Repeat the activity with a different triangle. Are the same things true?



Theorem

Triangle Midsegment Theorem

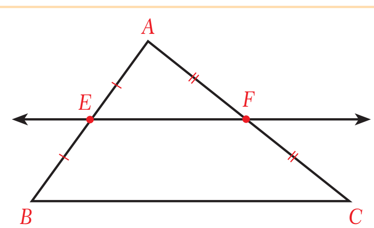
The line segment which joins the midpoints of two sides of a triangle is called a **midsegment** of the triangle. It is parallel to the third side and its length is equal to half the length of the third side.

Proof Let us draw an appropriate figure.

Given: $AE = EB$ and $AF = FC$

Prove: $EF \parallel BC$ and $EF = \dfrac{BC}{2}$

Let us begin by drawing a line through E parallel to the line BC.

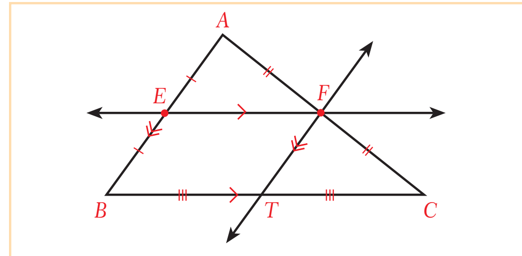


By the previous theorem, this line will pass through the midpoint F of the side AC.

So $EF \parallel BC$.

Now let us draw a line parallel to AB which passes through F.

By the previous theorem, it passes through the midpoint T of the side BC. Now,



$\angle BAC \cong \angle TFC$	(Corresponding angles)
$AF \cong FC$	(Given)
$\angle AFE \cong \angle FCT$.	(Corresponding angles)

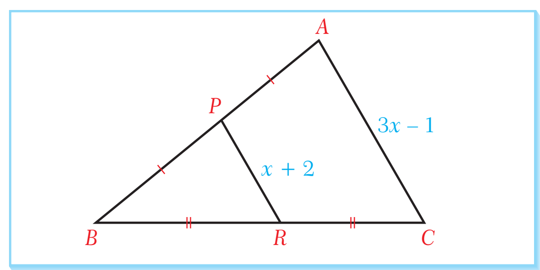
So by the ASA Congruence Theorem, $\Delta EAF \cong \Delta TFC$, and so $EF = TC$ because CPCTC.

Also, since T is the midpoint of BC, $TC = \dfrac{BC}{2}$.

So $EF = TC = \dfrac{BC}{2}$, which completes the proof.

EXAMPLE 82 In a triangle ABC, P and R are the midpoints of AB and BC, respectively. $AC = 3x - 1$ and $PR = x + 2$ are given. Find PR.

Solution



- $PR = \dfrac{1}{2}AC$ (Triangle Midsegment Theorem)
- $x + 2 = \dfrac{1}{2} \cdot (3x - 1)$ (Substitute)
- $x = 5$ (Simplify)

So $PR = 5 + 2 = 7$.

Theorem

Angle Bisector Theorem

The distances from a point lying on the bisector of an angle to each side of the angle are congruent.

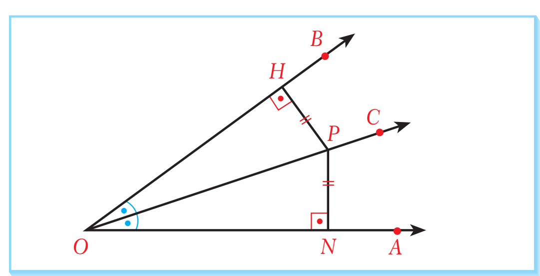
Proof Look at the figure.

Given: $\angle BOC \cong \angle AOC$,

$PH \perp OB$ and

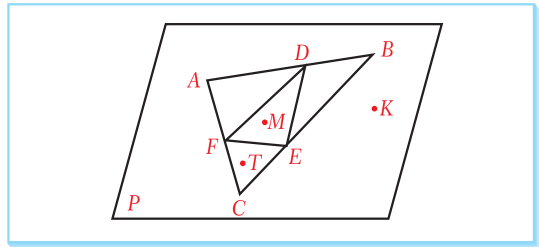
$PN \perp OA$

Prove: $PH \cong PN$



The distance from a point A to a line m is the length of the line segment AB such that $B \in m$ and $AB \perp m$.

EXAMPLE 83 Write whether each statement is true or false according to the figure opposite.



a. Point T is in the interior of ΔDFE.

b. $M \in$ ext ΔBDE

c. $\Delta ADF \cap \Delta BED = \varnothing$

d. ext $\Delta FDE \cap$ int $\Delta FCE = \Delta FCE$

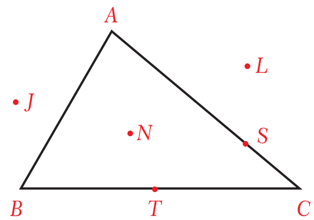
e. Points T and K are in the exterior of ΔDFE.

Solution a. false b. true c. false d. false e. true

A physical model of a triangle with its interior region

Check Yourself 2

Answer according to the figure.



a. Name five points which are on the triangle.

b. Name three points which are not on the triangle.

c. Name two points which are in the exterior of the triangle.

d. What is the intersection of the line ST and the triangle ABC?

e. What is the intersection of the segment NS and the exterior of the triangle ABC?

Answers

a. points A, B, C, T and S b. points J, L and N c. points J and L d. points S and T

e. $\varnothing$

2. Auxiliary Elements of a Triangle

Auxiliary elements are extra or additional elements.

Three special line segments in a triangle can often help us to solve triangle problems. These segments are the median, the altitude and the bisector of a triangle.

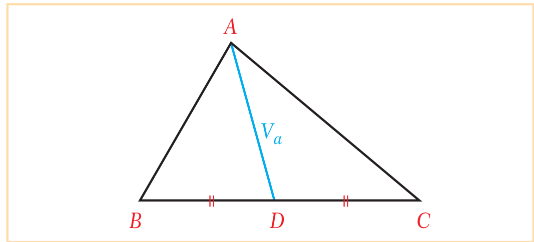
a. Median

Definition

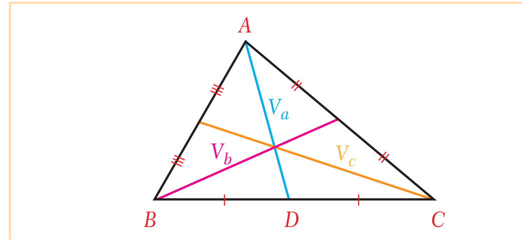
median

In a triangle, a line segment whose endpoints are a vertex and the midpoint of the side opposite the vertex is called a **median** of the triangle.

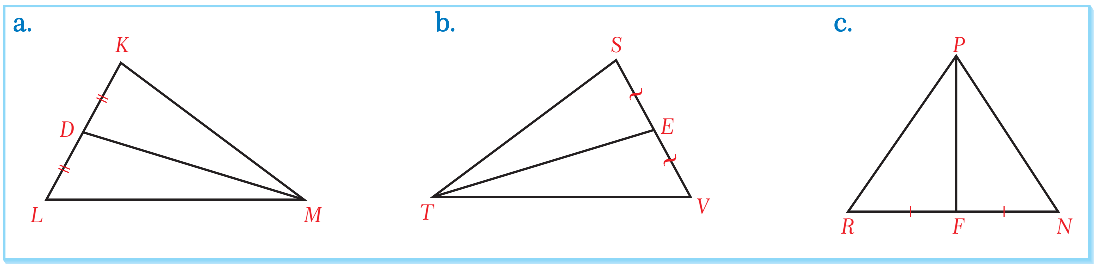
In the figure, the median to side BC is the line segment AD. It includes the vertex A and the midpoint of BC.



We usually use the capital letter V to indicate the length of a median. Accordingly, the lengths of the medians from the vertices of a triangle ABC to each side a, b and c are written as V_a, V_b and V_c, respectively. As we can see, every triangle has three medians.



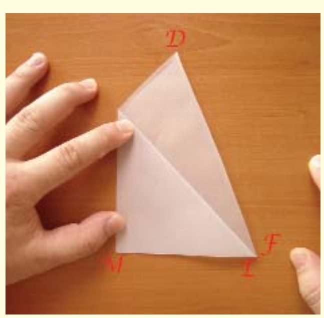
EXAMPLE 84 Name the median indicated in each triangle and indicate its length.



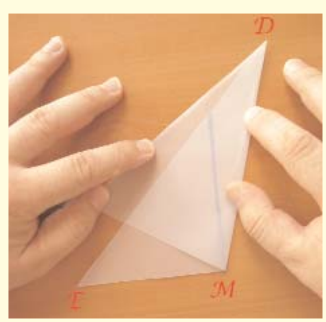
Solution
a. median MD, length V_m
b. median TE, length V_t
c. median PF, length V_p

Activity — *Paper Folding - Medians*

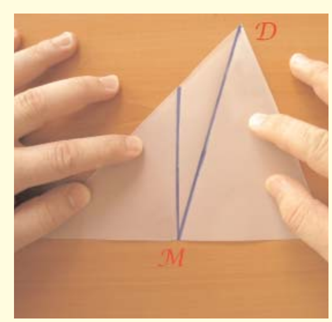
1. Follow the steps to construct the median of a triangle by paper folding.



Take a triangular piece of paper and fold one vertex to another vertex. This locates the midpoint of a side.



Fold the paper again from the midpoint to the opposite vertex.



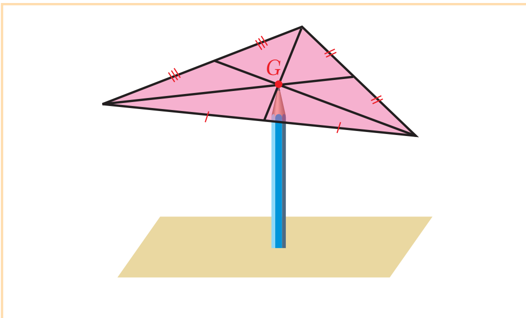
DM is the median of EF.

2. Cut out three different triangles. Fold the triangles carefully to construct the three medians of each triangle. Do you notice anything about how the medians of a triangle intersect each other?

> **Definition**
>
> **centroid of a triangle**
>
> The medians of a triangle are concurrent. Their common point is called the **centroid** of the triangle.

Concurrent lines are lines which all pass through a common point.

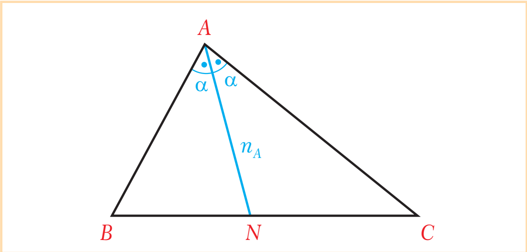
The centroid of a triangle is the center of gravity of the triangle. In other words, a triangular model of uniform thickness and density will balance on a support placed at the centroid of the triangle. The two figures below show a triangular model which balances on the tip of a pencil placed at its centroid.



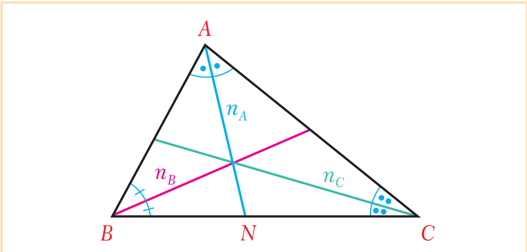
b. Angle bisector

> **Definition**
>
> **triangle angle bisector**
>
> An **angle bisector** of a triangle is a line segment which bisects an angle of the triangle and which has an endpoint on the side opposite the angle.

In the figure, AN is the angle bisector which divides $\angle BAC$ into two congruent parts. We call this the bisector of angle A because it extends from the vertex A. Since AN is an angle bisector, we can write $m(\angle BAN) = m(\angle NAC)$.

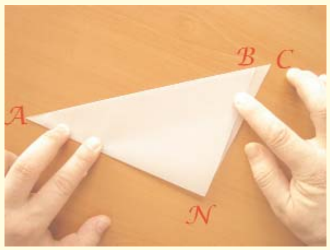


We usually use the letter n to indicate the length of an angle bisector in a triangle. Hence the lengths of the angle bisectors of a triangle ABC from vertices A, B and C are written n_A, n_B and n_C, respectively. As we can see, every triangle has three angle bisectors.

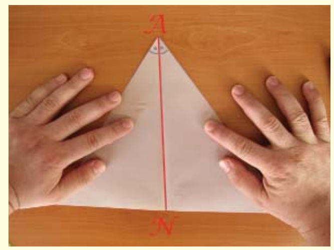


Follow the steps to explore angle bisectors in a triangle.

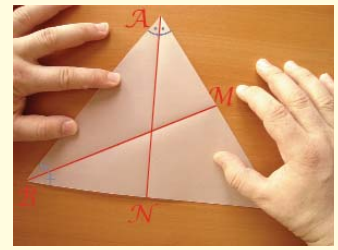
1. Cut out three different triangles.
2. Fold the three angle bisectors of each triangle as shown below.
3. What can you say about the intersection of the angle bisectors in a triangle?



Folding the angle bisector of $\angle A$.



AN is the angle bisector of $\angle A$.



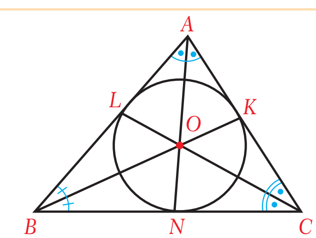
BM is the angle bisector of $\angle B$.

Definition

incenter of a triangle

The angle bisectors in a triangle are concurrent and their intersection point is called the **incenter** of the triangle. The incenter of a triangle is the center of the inscribed circle of the triangle.

The **inscribed circle** of a triangle is a circle which is tangent to all sides of the triangle.

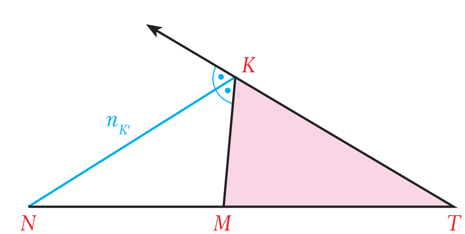


O is the incenter of ΔABC

As an exercise, try drawing a circle centered at the incenter of each of your triangles from the previous activity. Are your circles inscribed circles?

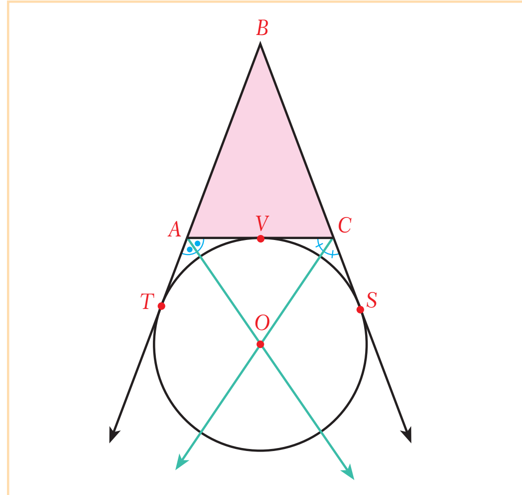
We have seen that n_A, n_B and n_C are the bisectors of the interior angles of a triangle ABC. We can call these bisectors **interior angle bisectors**. Additionally, the lengths of the bisectors of the exterior angles $\angle A'$, $\angle B'$ and $\angle C'$ are written as $n_{A'}$, $n_{B'}$ and $n_{C'}$ respectively. These bisectors are called the **exterior angle bisectors** of the triangle.

In the figure at the right, segment KN is the exterior angle bisector of the angle K' in ΔKMT and its length is $n_{K'}$.



Definition **excenter of a triangle**

The bisectors of any two exterior angles of a triangle are concurrent. Their intersection is called an **excenter** of the triangle.

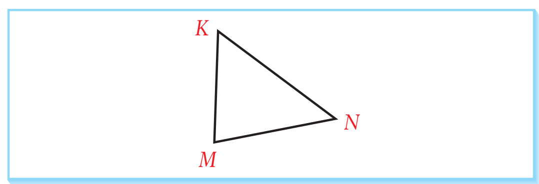


In the figure, ABC is a triangle and the bisectors of the exterior angles A' and C' intersect at the point O. So O is an excenter of ΔABC. In addition, O is the center of a circle which is tangent to side AC of the triangle and the extensions of sides AB and BC of the triangle. This circle is called an **escribed circle** of ΔABC.

An **escribed circle** of a triangle is a circle which is tangent to one side of the triangle and the extensions of the other two sides.

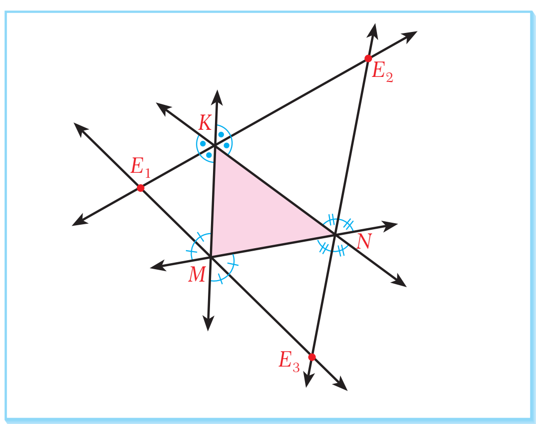
As we can see, a triangle has three excenters and three corresponding escribed circles.

EXAMPLE 85 Find all the excenters of ΔKMN in the figure by construction.



Solution To find the excenters, we first construct the bisector of each exterior angle using the method we learned in Chapter 1. Then we use a straightedge to extend the bisectors until they intersect each other.

The intersection points E_1, E_2 and E_3 are the excenters of ΔKMN.



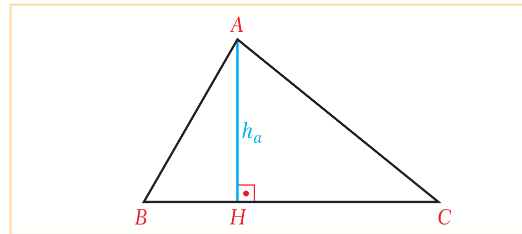
c. Altitude

Definition

altitude of a triangle

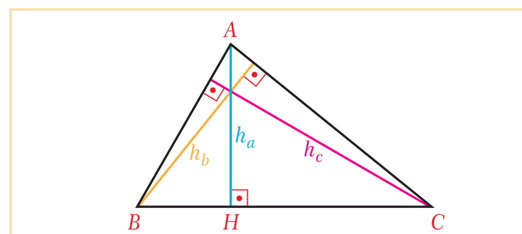
An **altitude** of a triangle is a perpendicular line segment from a vertex of the triangle to the line containing the opposite side of the triangle.

In the figure, AH is the altitude to side BC because AH is perpendicular to BC.

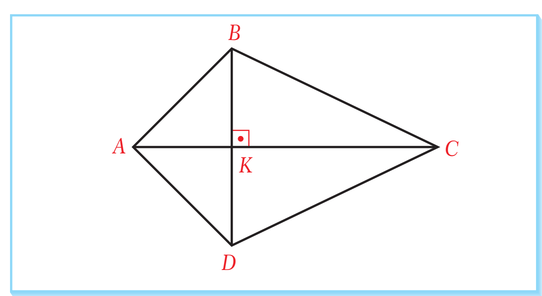


In a triangle, the length of an altitude is called a **height** of the triangle.

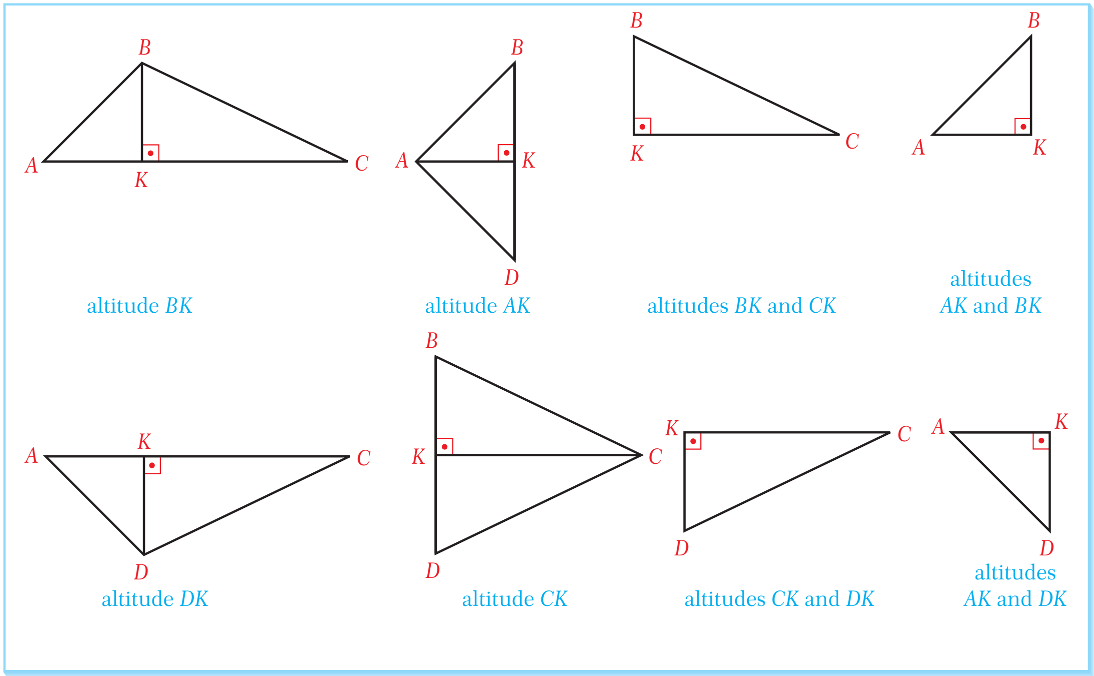
The heights from sides a, b and c of a triangle ABC are usually written as h_a, h_b and h_c, respectively. As we can see, every triangle has three altitudes.



EXAMPLE 86 Name all the drawn altitudes of all the triangles in the figure.



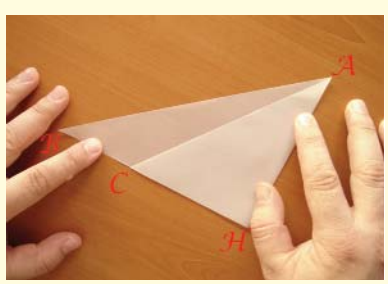
Solution There are eight triangles in the figure. Let us look at them one by one and name the drawn altitudes in each.

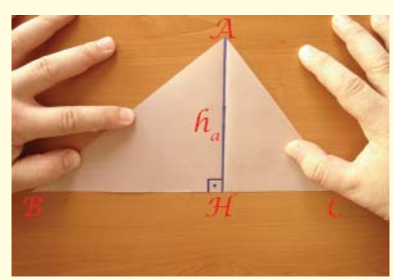


Activity

Paper Folding - Altitudes

To fold an altitude, we fold a triangle so that a side matches up with itself and the fold contains the vertex opposite the side.





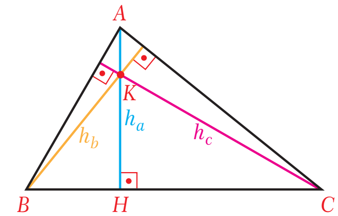
Cut out three different triangles. Fold them carefully to construct the three altitudes of each triangle. What can you say about how the altitudes intersect?

Definition

orthocenter of a triangle

The altitudes of a triangle are concurrent. Their common point is called **orthocenter** of the triangle.

Since the position of the altitudes of a triangle depends on the type of triangle, the position of the orthocenter relative to the triangle changes. In the figure opposite, the orthocenter K is in the interior region of the triangle. Later in this chapter we will look at two other possible positions for the orthocenter.



K is the orthocenter of ΔABC

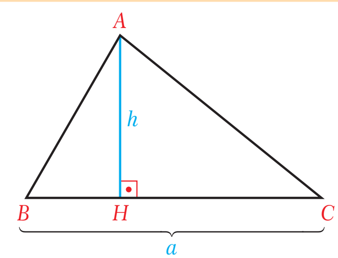
Once we know how to draw an altitude of a triangle, we can use it to find the area of the triangle.

Definition

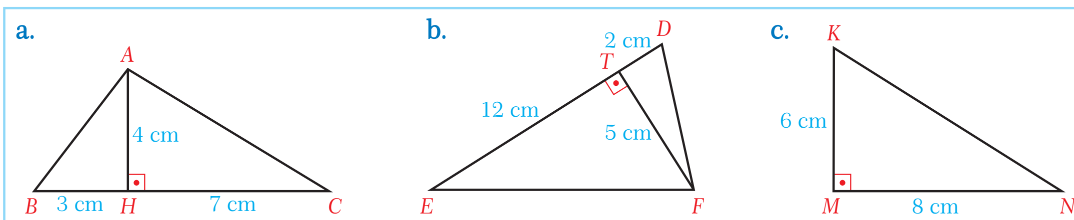
area of a triangle

The area of a triangle is half the product of the length of a side (called the base of the triangle) and the height of the altitude drawn to that base. We write $\boldsymbol{A(\Delta ABC)}$ to mean the area of ΔABC.

For example, the area of ΔABC in the figure is $A(\Delta ABC) = \dfrac{BC \cdot AH}{2} = \dfrac{a \cdot h}{2}$. Area is usually expressed in terms of a square unit.



EXAMPLE 87 Find the area of each triangle.



Solution a.

$$A(\Delta ABC) = \frac{BC \cdot AH}{2} \quad \text{(Definition of the area of a triangle)}$$

$$= \frac{10 \cdot 4}{2} \quad \text{(Substitute)}$$

$$= 20\ \text{cm}^2 \quad \text{(Simplify)}$$

b. $A(\Delta DEF) = \dfrac{FT \cdot DE}{2}$ (Definition of the area of a triangle)

$= \dfrac{5 \cdot 14}{2}$ (Substitute)

$= 35 \text{ cm}^2$ (Simplify)

c. $A(\Delta KMN) = \dfrac{KM \cdot MN}{2}$ (Definition of the area of a triangle)

$= \dfrac{6 \cdot 8}{2}$ (Substitute)

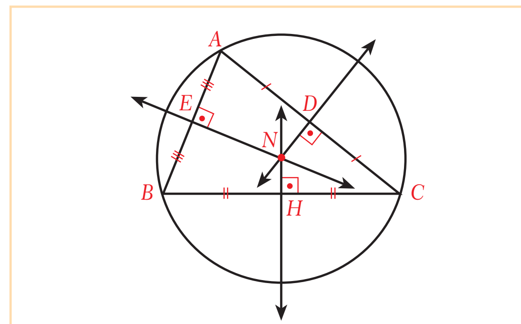
$= 24 \text{ cm}^2$ (Simplify)

Definition

perpendicular bisector of a triangle

In a triangle, a line that is perpendicular to a side of the triangle at its midpoint is called a **perpendicular bisector** of the triangle.

In the figure, HN, DN and EN are the perpendicular bisectors of triangle ABC. Perpendicular bisectors in a triangle are always concurrent.



The picture below hangs straight when the hook lies on the perpendicular bisector of the picture's top edge.

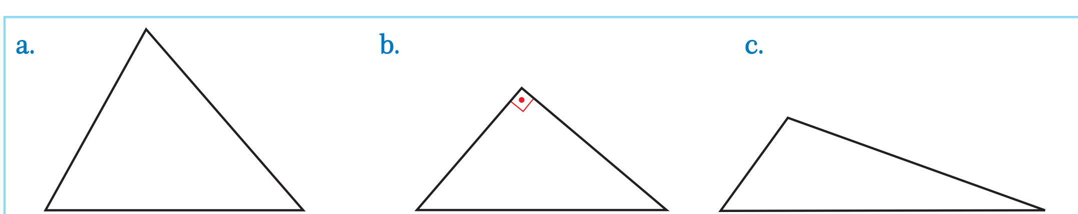
Definition

circumcenter of a triangle

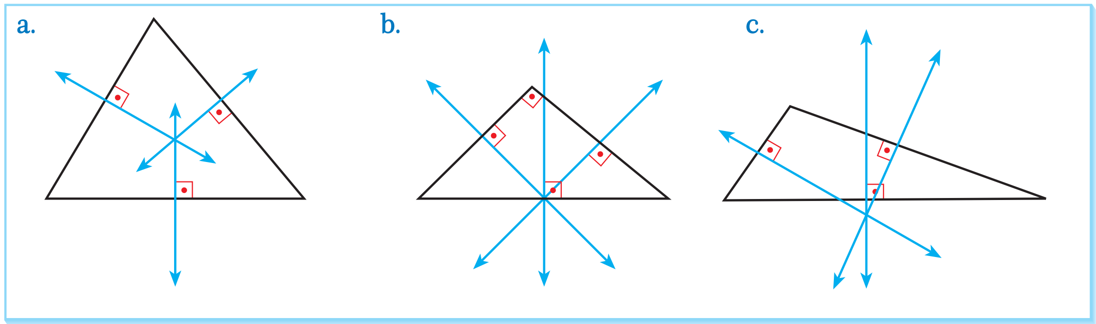
The intersection point of the perpendicular bisectors of a triangle is called the **circumcenter** of the triangle. The circumcenter of a triangle is the center of the circumscribed circle of the triangle.

The **circumscribed circle** of a triangle is a circle which passes through all the vertices of the triangle.

EXAMPLE 88 Find the circumcenter of each triangle by construction.



Solution First we construct the perpendicular bisector of each side of the triangle. Their intersection point is the circumcenter of the triangle.



Activity — *Perpendicular Bisector of a Triangle*

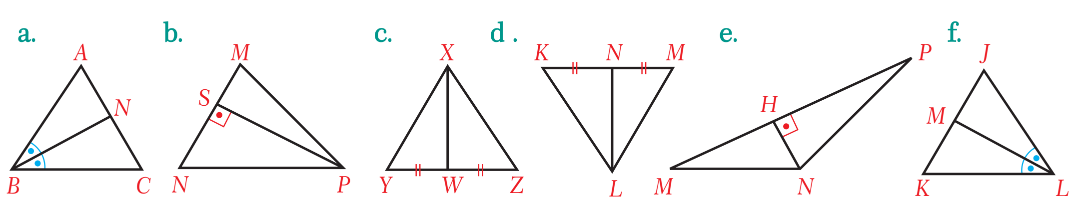
There are three main faculties on a university campus. The university wants to build a library on the campus so that it is the same distance from each faculty.

1. Make a geometric model of the problem.
2. Find the location of the library in the picture opposite.

As an exercise, draw three more triangles on a piece of paper and construct their circumcenters. Check that each circumcenter is the center of the inscribed circle.

Check Yourself 3

1. Name the auxiliary element shown in each triangle using a letter (n, h or V) and a vertex or side.



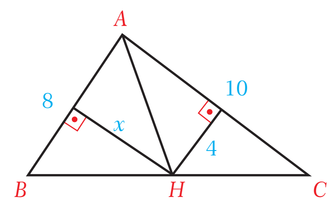
2. In a triangle MNP, the altitude NT of side MP and the median MK of side NP intersect at the point R.

   a. Name all the triangles in the figure formed. b. Name two altitudes of ΔMTN.

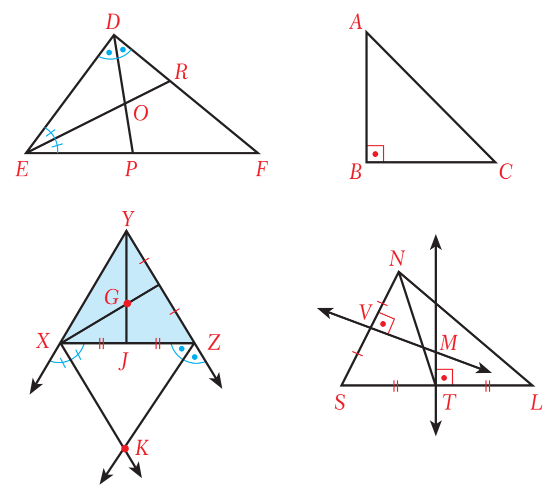
3. In a triangle DEF, EM is the median of side DF. If $DE = 11.4$, $MF = 4.6$ and the perimeter of ΔDEF is 27, find the length of side EF.

4. In a triangle KLM, LN is the altitude of the side KM. We draw the angle bisectors LE and LF of angles KLN and MLN respectively. If the angles between the angle bisectors and the altitude are 22° and 16° respectively, find $m(\angle KLM)$.

5. In the figure, $A(\Delta ABH) = A(\Delta AHC)$. Find x.



6. Write one word or letter in each gap to make true statements about the figures.



   a. Point O is a(n) __________ .
   b. Segment ________ is a median.
   c. Point _______ is an excenter.
   d. Segment ________ is an altitude.
   e. Point B is a(n) _____________.
   f. Segment ER is a(n) __________ __________.
   g. Point _________ is a circumcenter.
   h. Line TM is a(n) __________ __________.
   i. Point ________ is a centroid.

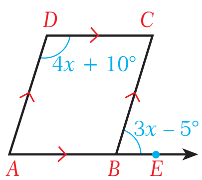
Answers

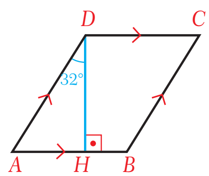
1. a. n_B b. h_p c. V_x d. V_l e. h_n f. n_L
2. a. ΔMNK, ΔMKP, ΔMNT, ΔNTP, ΔMRT, ΔMNR, ΔRNK, ΔMNP b. NT, TM
3. 6.4 4. 76° 5. 5
6. a. incenter b. ET c. K d. AB (or BC) e. orthocenter (or vertex) f. angle bisector g. M h. perpendicular bisector i. G

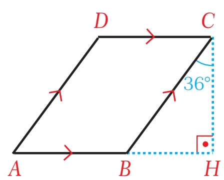
EXERCISES 1.1

A. Parallelogram

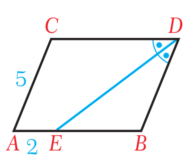
1. Find the measures of the interior angles of each parallelogram, using the information given.

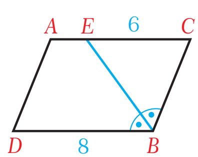
a.


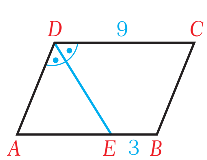
b.


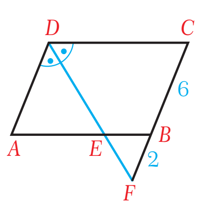
c.


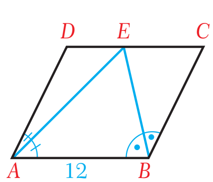
2. Each figure below shows a parallelogram with one or more angle bisectors. Find the perimeter of each parallelogram, using the information given.

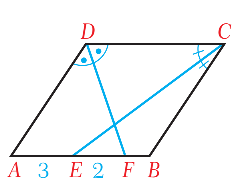
a.


b.


c.


d.


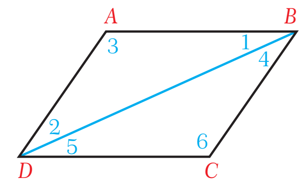
e.


f.


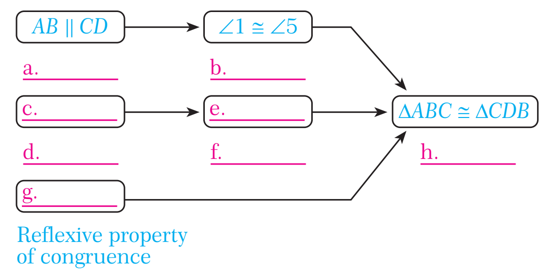
3. Complete the flow chart to prove that a diagonal of a parallelogram divides the paralellogram into two congruent triangles.

Given: parallelogram $ABCD$ with diagonal BD

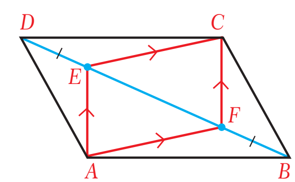
Prove: $\Delta ABD \cong \Delta CDB$



Proof:



4. In the figure, $AFCE$ is a parallelogram, $DE = BF$ and points B, F, E and D are collinear. Prove that quadrilateral $ABCD$ is a parallelogram.

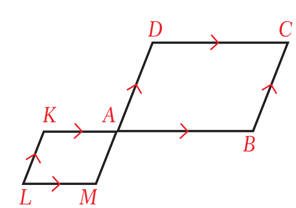


5. In the figure below, quadrilaterals $KLMA$ and $ABCD$ are parallelograms. Points K, A and B are collinear, and points M, A and D are also collinear. Prove each statement by using either a paragraph proof, a flow chart proof, or a two-column proof.

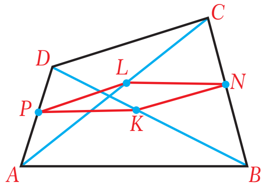
a. $\angle L \cong \angle C$

b. $LM \parallel DC$

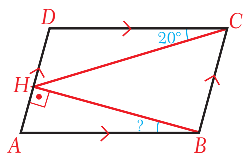
c. $\angle K$ and $\angle C$ are supplementary



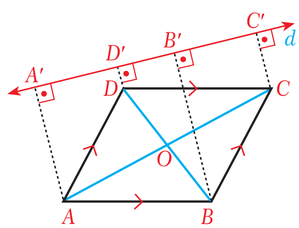
6. In the figure, AC and BD are diagonals of the quadrilateral $ABCD$. Points P and N are the midpoints of sides AD and BC respectively, and points L and K are the midpoints of diagonals AC and BD respectively. Prove that quadrilateral $PKNL$ is a parallelogram.



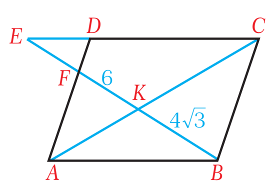
7. In the figure, $ABCD$ is a parallelogram with $BH \perp AD$ and $BH = AD$.
If $m(\angle HCD) = 20°$, find $m(\angle ABH)$.



8. In the figure, line d and parallelogram $ABCD$ have no common points and d is perpendicular to each of AA_1, BB_1, CC_1 and DD_1. If $AA_1 = 7$ cm, $BB_1 = 9$ cm and $DD_1 = 3$ cm, find the length of CC_1.



9. In the figure, $ABCD$ is a parallelogram. Points B, K, F, E and C, D, E are respectively collinear, and AC is the diagonal of the parallelogram. Given $BK = 4\sqrt{3}$ cm and $FK = 6$ cm, find the length of EF.

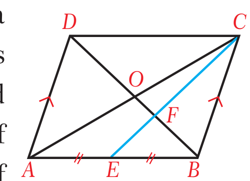


10. A parallelogram $ABCD$ has side lengths a and b and diagonals with lengths e and f. If $a + b = 13$ cm and $a \cdot b = 36$ cm, find the value of $e^2 + f^2$.

11. $ABCD$ is a parallelogram with $AB > BC$. Point E is on side BC such that $CE : EB = 3 : 1$, and point F is the intersection of DE and AC. If $AF = 8$ cm and $FE = 4$ cm, find the sum of the lengths of DF and FC.

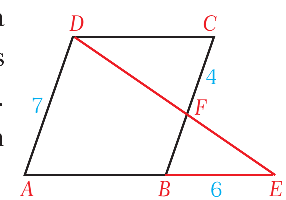
12. $ABCD$ is a parallelogram and points E and F are the midpoints of sides BC and CD respectively. AF and AE intersect the diagonal BD at points M and N respectively. Prove that $DM = MN = NB$.

13. In the figure, $ABCD$ is a parallelogram, point E is midpoint of side AB and point O is the intersection of the diagonals AC and BD. If $OF = 3$ cm, find



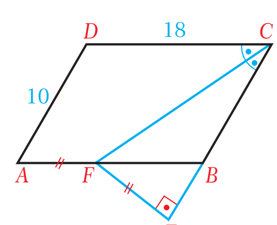
a. the ratio $\dfrac{FC}{FE}$. **b.** the length of BD.

14. In the figure, $ABCD$ is a parallelogram and points A, B and E are collinear. Point F is the intersection of DE and BC.

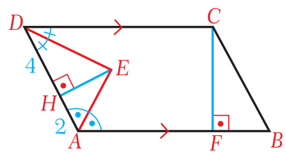


If $AD = 7$ cm, $BE = 6$ cm and $FC = 4$ cm, find $P(ABCD)$.

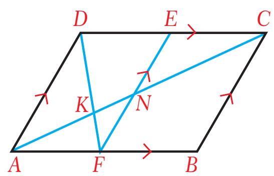
15. ✪ In the figure, $ABCD$ is a parallelogram and CF bisects $\angle C$. $FE \perp EC$, $AF = FE$, $AD = 10$ cm and $DC = 18$ cm are given. Find the perimeter of the right triangle EBF.



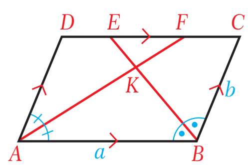
16. In the figure, $ABCD$ is a parallelogram and DE and AE bisect $\angle D$ and $\angle A$ respectively. If $EH \perp AD$, $AH = 2$ cm and $DH = 4$ cm, find the length of CF.



17. In the figure, $ABCD$ is a parallelogram such that $2 \cdot DE = 3 \cdot EC$, $EF \parallel BC$ and $AC = 40$ cm. Find the length of KN.



18. In the figure, $ABCD$ is a parallelogram and AF and BE are the bisectors of $\angle A$ and $\angle B$ respectively. If $AB = a$ and $BC = b$, show that $EF = 2b - a$.



B. Rectangle

19. One side of a rectangle measures 12 cm and its diagonal measures 13 cm. Find the perimeter of this rectangle.

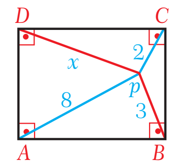
20. The length of the longer side of a rectangle is twice the length of its shorter side. If the perimeter of the rectangle is 36 cm, find

a. the lengths of the sides of the rectangle.

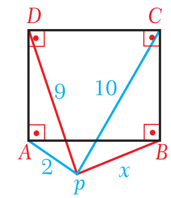
b. the length of the diagonal.

21. Calculate the length x in each figure.

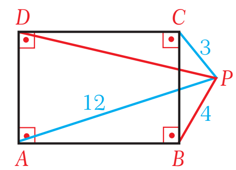
a.



b.



c.

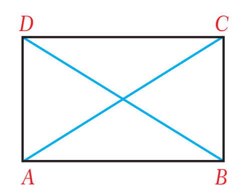


$|DP| = x$

22. Complete the paragraph to prove that if the diagonals of a parallelogram are congruent then the parallelogram is a rectangle.

Given: $ABCD$ is parallelogram and $DB = CA$.

Prove: $ABCD$ is a rectangle.

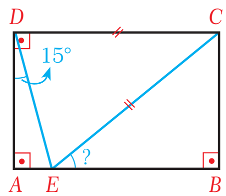


Proof:

Opposite sides of a parallelogram are congruent, so $DA \cong$ a.__________. Also, $AB \cong BA$ by the reflexive property of congruence. Since $DB = CA$ (given), $\Delta DAB \cong \Delta CBA$ by b.____________ . $\angle DAB \cong \angle CBA$ because c.____________ , and $\angle DAB$ and $\angle CBA$ are supplementary because they are d.____________ angles.

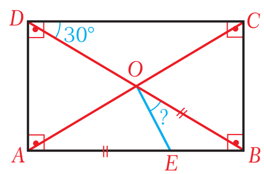
$\angle DAB$ and $\angle CBA$ are right angles because e.____________ . Hence $\angle CDA$ and $\angle DCB$ are also right angles because f.____________ . So $ABCD$ is a rectangle by g.____________ .

23. In the figure, $ABCD$ is a rectangle. If $CD = EC$ and $m(\angle ADE) = 15°$, find the measure of $\angle CEB$.



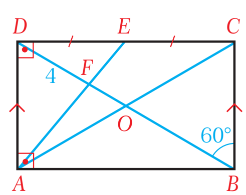
24. In the figure, $ABCD$ is a rectangle and point O is the intersection of the diagonals AC and BD.

If $AE = OB$ and $m(\angle BDC) = 30°$, find $m(\angle EOB)$.



25. $ABCD$ is a rectangle with $AB > BC$. H is a point on diagonal AC, and BH is perpendicular to AC. BH also divides AC into two line segments with lengths 9 cm and 16 cm. Find the perimeter of $ABCD$.

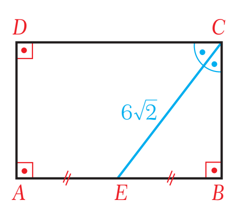
26. In the figure, point O is the intersection of the diagonals of the rectangle $ABCD$. If $DE = EC$, $DF = 4$ cm and $m(\angle OBC) = 60°$, find



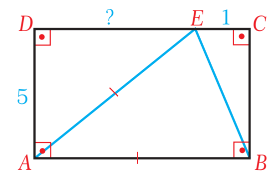
a. the length of the diagonal.

b. the lengths of sides.

27. In the rectangle $ABCD$ in the figure, CE is the bisector of $\angle C$ and point E is the midpoint of side AB. If $CE = 6\sqrt{2}$ cm, find the perimeter of the rectangle.

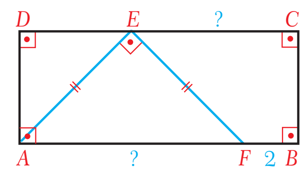


28. In a rectangle $ABCD$, point E is on side DC, $AB = AE$, $AD = 5$ cm and $EC = 1$ cm. Find the length of DE.

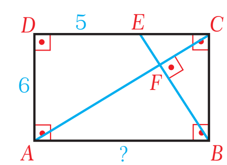


29. In a rectangle $ABCD$, points E and F are on sides DC and AB respectively.

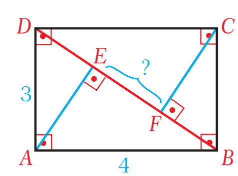
If $FB = 2$ cm and $BC = 5$ cm, find the lengths of AF and EC.



30. In a rectangle $ABCD$, point E is on side DC and point F is the intersection of BE and the diagonal AC. If $AC \perp BE$, $DE = 5$ cm and $AD = 6$ cm, find the length of AB.

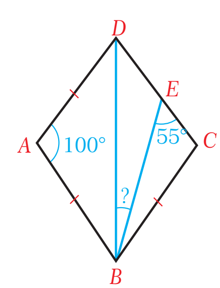


31. In a rectangle $ABCD$, points E and F are on the diagonal DB. Given $AE \perp DB$, $CF \perp DB$, $AB = 4$ cm and $AD = 3$ cm, find the length of EF.

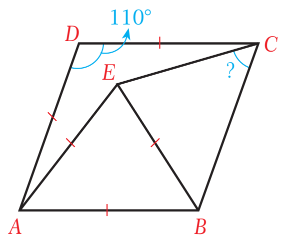


C. Rhombus

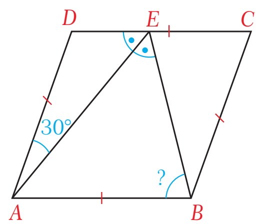
32. In the figure, $ABCD$ is a rhombus with $m(\angle A) = 100°$ and $m(\angle BEC) = 55°$. Find $m(\angle DBE)$.



33. In the figure, $ABCD$ is a rhombus and ABE is an equilateral triangle. If $m(\angle D) = 110°$, find $m(\angle BCE)$.

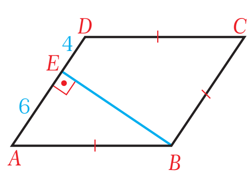


34. In the figure, $ABCD$ is a rhombus, point E is on side DC and AE is the bisector of $\angle DEB$. If $m(\angle DAE) = 30°$, find $m(ABE)$.



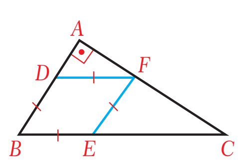
35. Find the perimeter of a rhombus whose diagonals measure 24 cm and 32 cm.

36. Quadrilateral $ABCD$ in the figure is a rhombus and BE is perpendicular to AD. If $AE = 6$ cm and $ED = 4$ cm, find the lengths of diagonals of the rhombus.

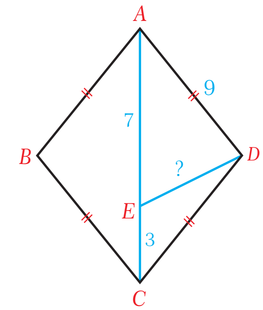


37. In the figure, ΔABC is a right triangle and the quadrilateral $BEFD$ is a rhombus.

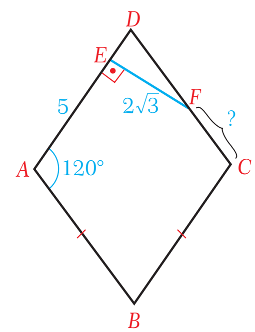
If $AB = 5$ cm and $AC = 12$ cm, find the length of one side of the rhombus.



38. In the figure, $ABCD$ is a rhombus and point E is on the diagonal AC.
If $AE = 7$ cm, $EC = 3$ cm and $AD = 9$ cm, find the length of DE.

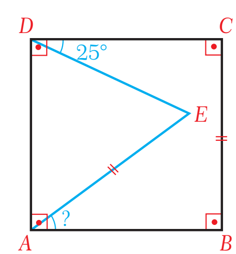


39. In the figure, $ABCD$ is a rhombus. Given $FE \perp AD$, $m(\angle DAB) = 120°$, $EA = 5$ cm and $EF = 2\sqrt{3}$ cm, find the length of FC.

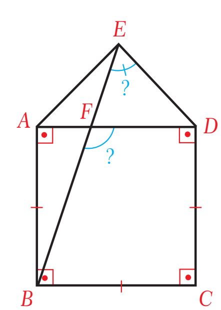


D. Square

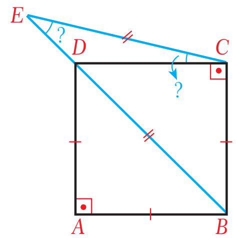
40. In the figure, $ABCD$ is a square with $AE = BC$ and $m(\angle CDE) = 25°$.
Find $m(\angle EAB)$.



41. In the figure, $ABCD$ is a square and ΔADE is equilateral. Find $m(\angle BFD)$ and $m(\angle BED)$.



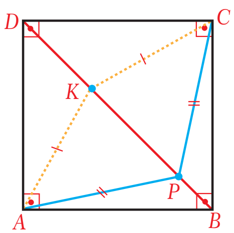
42. In the figure, $ABCD$ is a square, points B, D and E are collinear, and $BD = EC$. Find $m(\angle DEC)$ and $m(\angle DCE)$.



43. Complete the two-column proof to prove that in a square, any point taken on a diagonal is equidistant from the vertices on either side of the diagonal.

Given: $ABCD$ is a square and point P is on diagonal DB.

Prove: $AP = CP$

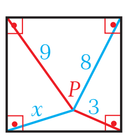


Proof:

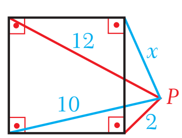
Statements	Reasons
$AB \cong CB$	**a.**
$\angle ABP \cong$ **b.**	**c.**
$BP \cong BP$	reflexive property of congruence
d. $\cong$ **e.**	SAS congruence postulate
$AP \cong CP$	**f.**
$AP = CP$	**g.**

44. In the following squares, P is any point. Calculate x in each figure, using the given lengths.

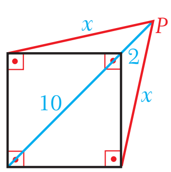
a.



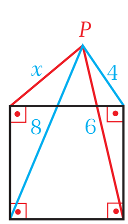
b.



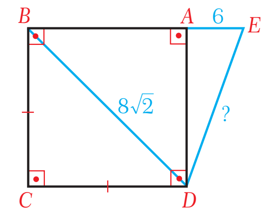
c.



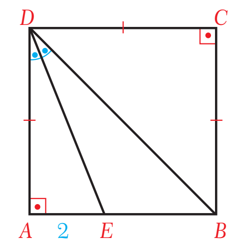
d.



45. In the figure, $ABCD$ is a square, $AE = 6$ cm and $BD = 8\sqrt{2}$ cm. Find the length of DE.

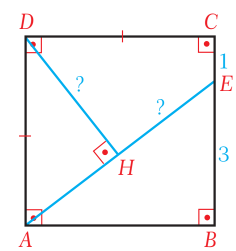


46. In the figure, $ABCD$ is a square and DE bisects $\angle ADB$. If $AE = 2$ cm, find the perimeter of the square.



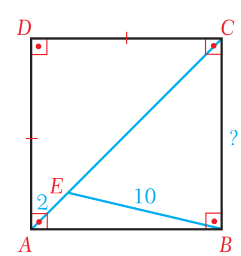
47. In a square $ABCD$, point E is on side BC and DH is perpendicular to AE.

If $BE = 3$ cm and $CE = 1$ cm, find the lengths of DH and HE.



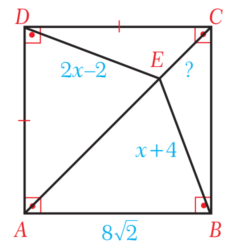
48. In a square $ABCD$, point E is on the diagonal AC.

If $AE = 2$ cm and $BE = 10$ cm, find the length of one side of the square.

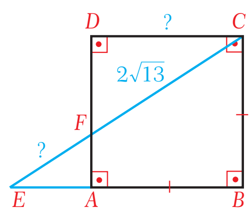


49. In the figure, $ABCD$ is a square and and point E is on the diagonal AC.

If $DE = 2x - 2$, $EB = x + 4$ and $AB = 8\sqrt{2}$ cm, find the length of EC.

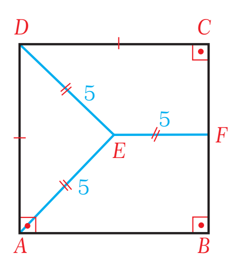


50. ✪ In the square $ABCD$ in the figure, points C, F, E and B, A, E are respectively collinear. Given that $DF = 2FA$ and $FC = 2\sqrt{13}$, find the lengths of EF and

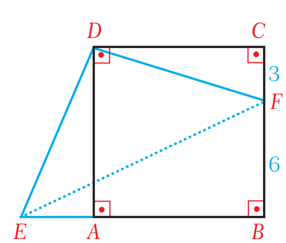


51. ✪ In the figure, $ABCD$ is a square and $EF \parallel AB$.

If $AE = DE = EF = 5$ cm, find the perimeter of the square.



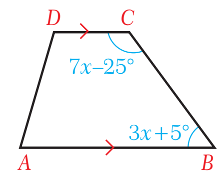
52. ✪✪ In the figure, $ABCD$ is a square and points E, A and B are collinear. Given $m(\angle EDF) = 90°$, $CF = 3$ cm and $BF = 6$ cm, find the length of EF.



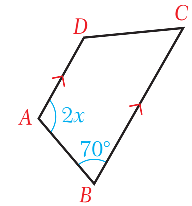
E. Trapezoid

53. In the following trapezoids, the bases are shown by parallel lines. Calculate x in each figure.

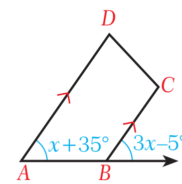
a.



b.

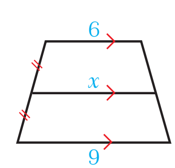


c.

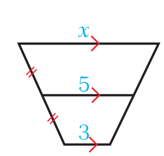


54. Each figure shows the lengths of the bases and the median of a trapezoid. Calculate x in each case.

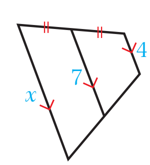
a.



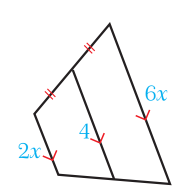
b.



c.

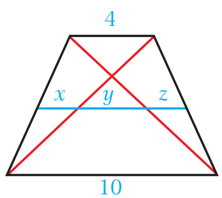


d.

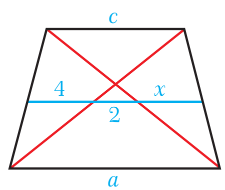


55. Each figure shows the median and diagonals of a trapezoid. Find the unknown lengths in each case, using the information given.

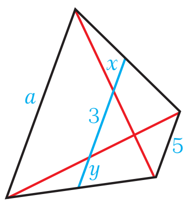
a.



b.

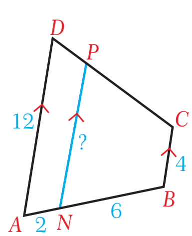


c.

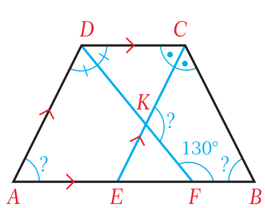


56. The median of a trapezoid divides the trapezoid into two new trapezoids. If the lengths of the medians of the new trapezoids are 8 cm and 12 cm, find the lengths of the bases of the original trapezoid.

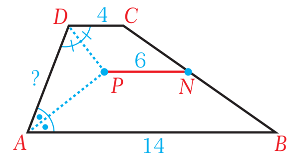
57. In the figure, $AD \parallel BC \parallel PN$. Given that $AD = 12$ cm, $BC = 4$ cm, $AN = 2$ cm and $NB = 6$ cm, find the length of PN.



58. In a trapezoid $ABCD$, $AB \parallel DC$ and DF and CE bisect $\angle D$ and $\angle C$ respectively. If $AD \parallel CE$ and $m(\angle DFB) = 130°$, find the measures of $\angle A$, $\angle B$ and $\angle CKF$.



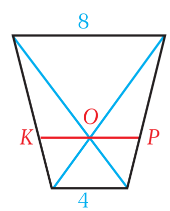
59. In the quadrilateral $ABCD$ in the figure, $AB \parallel PN \parallel DC$, and AP and DP are the bisectors of $\angle A$ and $\angle D$ respectively.



If $DC = 4$ cm, $PN = 6$ cm and $AB = 14$ cm, find the length of AD.

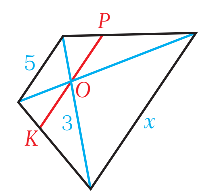
60. In each trapezoid below, point O is the intersection of the diagonals and KP is the line segment passing through this point which is parallel to the bases. Find x in each figure.

a.

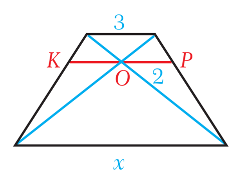


$KP = x$

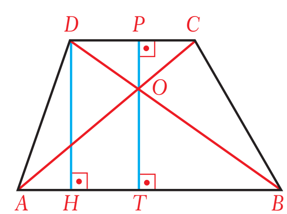
b.



c.



61. In the trapezoid opposite, $AB \parallel DC$, $DH \perp AB$, $PT \perp AB$ and points P, O and T are collinear.

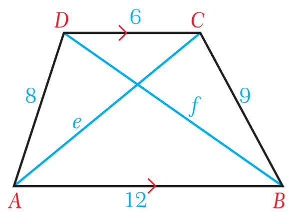


If $AB = 12$ cm, $DC = 4$ cm and $DH = 6$ cm, find the lengths of OP and OT.

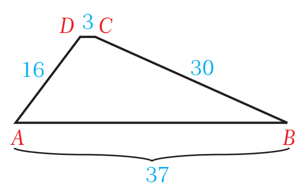
62. ✪✪ $ABCD$ is a trapezoid with $AB \parallel DC$.
Prove that $AC^2 + DB^2 = BC^2 + AD^2 + 2 \cdot AB \cdot DC$.

63. In a trapezoid $ABCD$, $AB \parallel DC$ and AC and BD are diagonals. $AB = 12$ cm, $BC = 9$ cm, $AD = 8$ cm and $DC = 6$ cm are given. If $AC = e$ and $BD = f$, find the value of $e^2 + f^2$.

(Hint: Use the formula from question 62.)



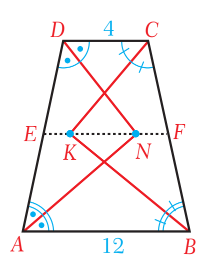
64. In a trapezoid $ABCD$, $AB \parallel DC$.
Given $AB = 37$ cm, $BC = 30$ cm, $AD = 16$ cm and $DC = 3$ cm, find the height of the trapezoid.



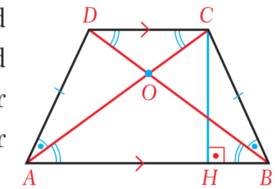
65. ✪ Prove that in a trapezoid, the angle formed by the bisectors of any two interior angles that share the same leg is a right angle, and the intersection of these bisectors lies on the median of the trapezoid.

66. ✪ In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$. AN, BK, CK and DN are the bisectors of $\angle A$, $\angle B$, $\angle C$ and $\angle D$ respectively.
Given $AB = 12$ cm, $DC = 4$ cm, $BC = 10$ cm and $AD = 8$ cm, find the length of KN.

(Hint: Use the theorem from question 65.)



67. The figure shows a trapezoid $ABCD$ with $AB \parallel DC$ and $AD = BC$. Decide whether each statement is true or false.



a. $ABCD$ is an isosceles trapezoid.

b. $\angle A \cong \angle B$ and $\angle D \cong \angle C$

c. $m(\angle A) + m(\angle D) = 180°$ and
$m(\angle A) + m(\angle C) = 180°$

d. $AC = DB$

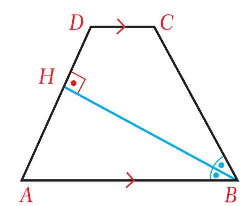
e. $DO = OC$ and $AO = BO$

f. $\angle OAB \cong \angle OBA$ and $\angle DAO \cong \angle CBO$

g. $\Delta ADB \cong \Delta BCA$, $\Delta AOD \cong \Delta BOC$ and $\Delta ACD \cong \Delta BCD$

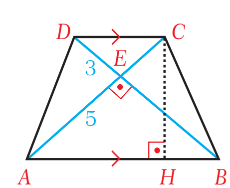
h. $HB = \dfrac{AB - DC}{2}$

68. In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$. If BH is the bisector of angle B and $BH \perp AD$, find the measures of the interior angles of the trapezoid.



69. ✪ Prove that if the diagonals of an isosceles trapezoid are perpendicular to each other, then the height of the trapezoid is equal to half the sum of the lengths of the bases.

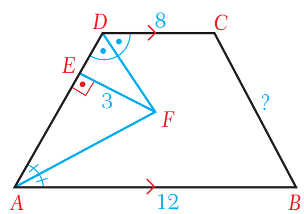
70. In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$ and $AC \perp BD$.
If $DE = 3$ cm and $AE = 5$ cm, find the length of CH.



71. In an isosceles trapezoid with base lengths 13 cm and 5 cm, each diagonal is perpendicular to a leg. Find the height of this trapezoid.

72. Find the lengths of the diagonals of an isosceles trapezoid whose base lengths are 6 cm and 18 cm, given that one leg measures 10 cm.

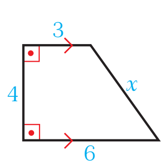
73. In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$, and AF and DF are bisectors of $\angle A$ and $\angle D$ respectively.



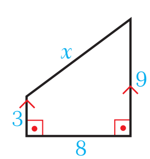
Given $EF \perp AD$, $EF = 3$ cm, $DC = 8$ cm and $AB = 12$ cm, find the length of BC.

74. Find the length x in each trapezoid.

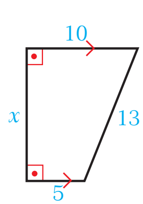
a.



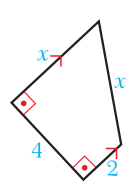
b.



c.

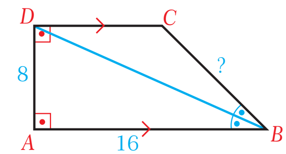


d.



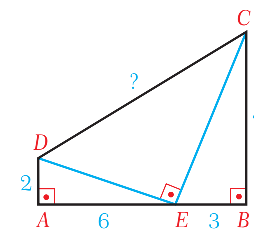
75. The diagonals of a right trapezoid are perpendicular to each other. Given that the bases measure 6 cm and 24 cm, find the height of the trapezoid.

76. In the figure, $ABCD$ is a right trapezoid with $AB \parallel DC$ and BD is the bisector of $\angle B$.



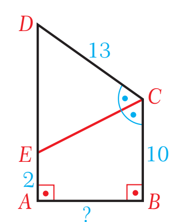
If $AD = 8$ cm and $AB = 16$ cm, find the length of BC.

77. In the right trapezoid $ABCD$ shown opposite, $AD \parallel BC$ and $DE \perp EC$. If $AD = 2$ cm, $AE = 6$ cm and $BE = 3$ cm, find the lengths of BC and DC.



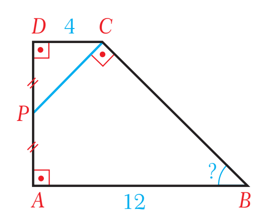
78. In the figure, $ABCD$ is a right trapezoid with $AD \parallel BC$, and CE bisects $\angle BCD$.

If $AE = 2$ cm, $DC = 13$ cm and $BC = 10$ cm, find the length of AB.



79. ✪ In the right trapezoid $ABCD$ in the figure, $AB \parallel DC$.

Given $AP = PD$, $PC \perp BC$, $DC = 4$ cm and $AB = 12$ cm, find

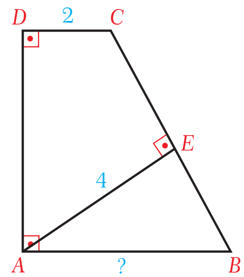


a. the length of AD.

b. the length of BC.

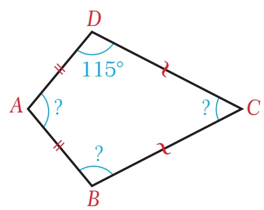
80. ✪✪ In the right trapezoid $ABCD$, $AB \parallel DC$ and $AE \perp BC$.

Given $AB = BC$, $DC = 2$ cm and $AE = 4$ cm, find the length of AB.

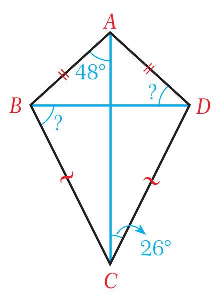


F. Kite

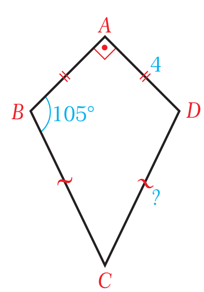
81. In the figure, $ABCD$ is a kite with $AB = AD$ and $CB = CD$.
If $m(\angle A) = 3 \cdot m(\angle C)$ and $m(\angle D) = 115°$, find the measures of $\angle A$, $\angle B$ and $\angle C$.



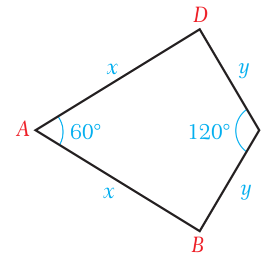
82. In the figure, $ABCD$ is a kite with $AB = AD$ and $CB = CD$.
If $m(\angle BAC) = 48°$ and $m(\angle ACD) = 26°$, find $m(\angle DBC)$ and $m(\angle BDA)$.



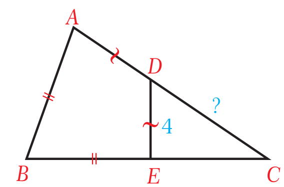
83. In the figure, $ABCD$ is a kite with $AB = AD$ and $CB = CD$. Given $m(\angle B) = 105°$ and $AD = 4$ cm, find the length of DC.



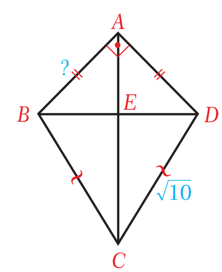
84. In the figure, $ABCD$ is a kite with $AB = AD = x$ and $CB = CD = y$.
$m(\angle A) = 60°$ and $m(\angle C) = 120°$ are given. Find the ratio of x to y.



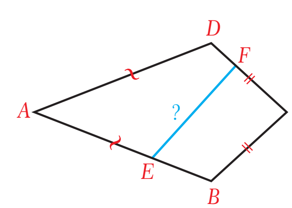
85. In the figure, $ABED$ is a kite with $AB = BE$ and $DE = 4$ cm.
If $3 \cdot BE = 2 \cdot EC$, find the length of DC.



86 ✪ In the figure, $ABCD$ is a kite with $AB = AD$ and $CB = CD$.
Given $m(\angle BAD) = 90°$, $CD = \sqrt{10}$ cm and $AC = 4$ cm, find the length of AB.



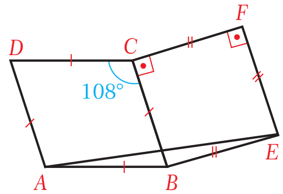
87. ✪✪ In the figure, $ABCD$ is a kite with $AB = AD$ and $CB = CD$. A, E, B and D, F, C are respectively collinear, and the diagonals of the kite measure 9 cm and 6 cm. If $FC = 2 \cdot DF$ and $AE = 2 \cdot EB$, find the length of EF.



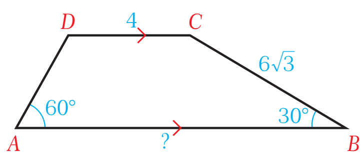
Mixed Problems

88. In the figure, $ABCD$ is a rhombus and $BEFC$ is a square.

If $m(\angle DCB) = 108°$, find $m(\angle AEF)$.

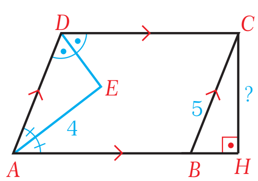


89.

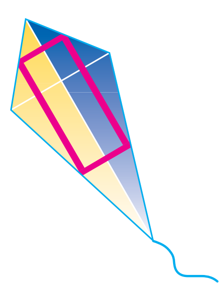


In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$, $BC = 6\sqrt{3}$ cm, $DC = 4$ cm, $m(\angle A) = 60°$ and $m(\angle B) = 30°$. Find the length of AB.

90. In the figure, $ABCD$ is a parallelogram and DE and AE are the bisectors of the $\angle D$ and $\angle A$ respectively. Points A, B and H are collinear. If $CH \perp AH$, $AE = 4$ cm and $BC = 5$ cm, find the length of CH.

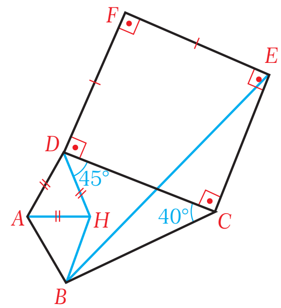


91. Beyza and Rana are designing a kite to look like the one at the right. Its diagonals will measure 44 cm and 60 cm, and the students will use ribbon to connect the midpoints of the sides. How much ribbon will Beyza and Rana need?

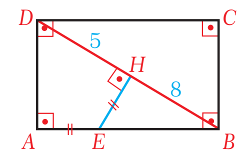


92. In the figure, $ABCD$ is a kite, ΔAHD is an equilateral triangle and $CEFD$ is a square.

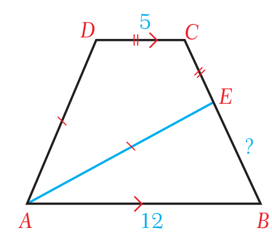
If $m(\angle BCD) = 40°$ and $m(\angle CDH) = 45°$, find $m(\angle HBE)$.



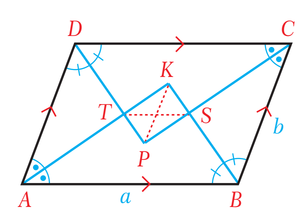
93. In the figure, $ABCD$ is a rectangle. Given $EH \perp BD$, $AE = EH$, $DH = 5$ cm and $HB = 8$ cm, find $P(AEHD)$.



94. In the figure, $ABCD$ is a trapezoid and $ADCE$ is a kite. Given $AB \parallel DC$, $DC = CE = 5$ cm and $AB = 12$ cm, find the length of EB.



95. In the figure, $ABCD$ is a parallelogram and AK, BK, CP and DP bisect $\angle A$, $\angle B$, $\angle C$ and $\angle D$ respectively. $AB = a$ and $BC = b$ are given.

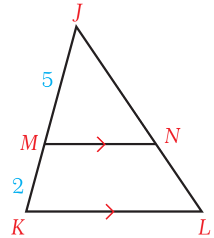


a. Prove that $TPSK$ is a rectangle.

b. Prove that $TS = PK = a - b$

C. The Triangle Proportionality Theorem and Thales' Theorem

13. In the figure, $MN \parallel KL$, $JM = 5$ and $MK = 2$. Find each ratio.

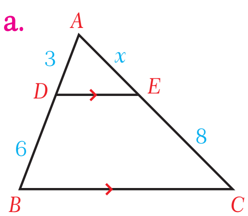


a. $\frac{JN}{NL}$ b. $\frac{JN}{JL}$

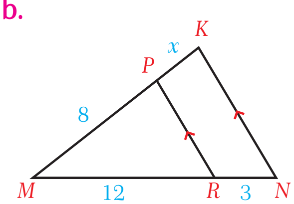
c. $\frac{NL}{JN}$ d. $\frac{NL}{JL}$

14. Find the value of x in each figure by using the information given.

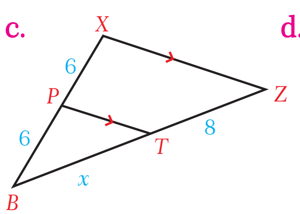
a.



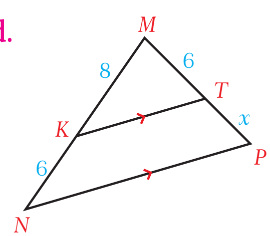
b.



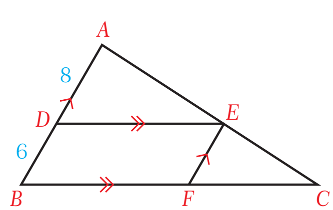
c.



d.

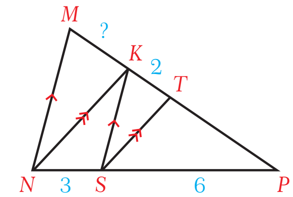


15. In the figure,
$AD = 8$ cm,
$DB = 6$ cm,
$EF \parallel AB$ and $DE \parallel BC$.
Find each ratio.

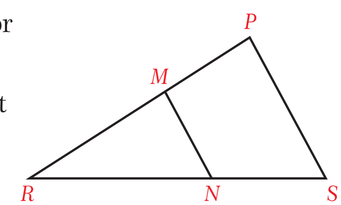


a. $\frac{AC}{AE}$ b. $\frac{AC}{EC}$ c. $\frac{BF}{FC}$ d. $\frac{FC}{BC}$

16. In the figure, MNP is a triangle and
$MN \parallel KS$,
$KN \parallel TS$,
$NS = 3$ cm,
$SP = 6$ cm and
$KT = 2$ cm.
What is the length of MK?



17. Determine whether or not $MN \parallel PS$ in the figure, given each set of extra information.



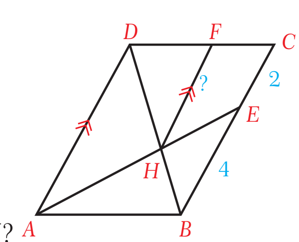
a. $PR = 18$ $MR = 6$
$SR = 24$ $NR = 8$

b. $PR = 12$ $MP = 8$
$SR = 16$ $NR = 12$

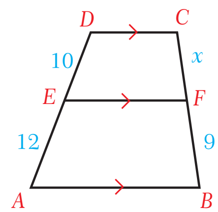
c. $MR = 5$ $MP = 4$
$RN = 6$ $NS = \frac{24}{5}$

d. $PR = 15$ $MR = 12$
$RN = 16$ $NS = 4$

18. $ABCD$ in the figure is a parallelogram with
$CE = 2$ cm,
$EB = 4$ cm and
$FH \parallel AD$.
What is the length of FH?



19. In the figure, $DC \parallel EF \parallel AB$. Find the value of x.

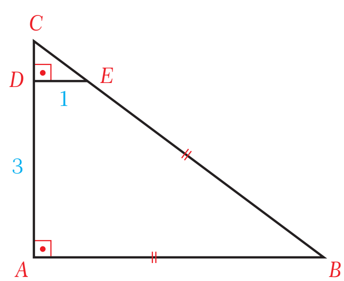


20. Write a proof of the Converse of the Triangle Proportionality Theorem in two-column form.

21. ✪ A point on the hypotenuse of a right triangle divides the hypotenuse into two segments of lengths 12 and 16. Given that the point is equidistant to the legs of the triangle, find the lengths of the legs of the triangle.

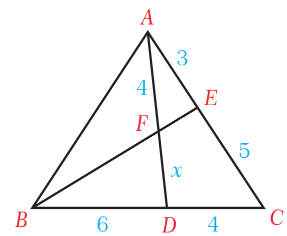
22. ✪ In the triangle ABC at the right, $m(\angle A) = 90°$, $CD \perp DE$, $DE = 1$, $AD = 3$ and $AB = BE$. Find the length of CE.

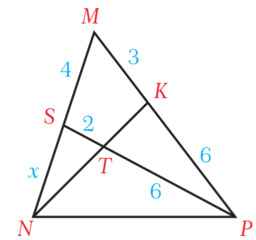
(Hint: Draw the perpendicular $EH \perp AB$.)

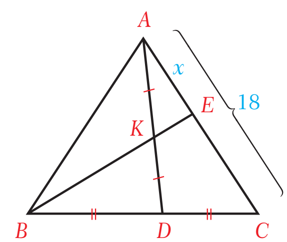


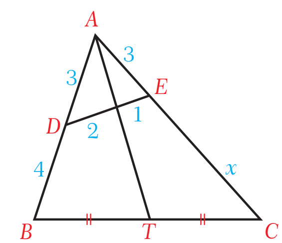
D. Further Applications

23. Find the length x in each figure by using Menelaus' Therom.





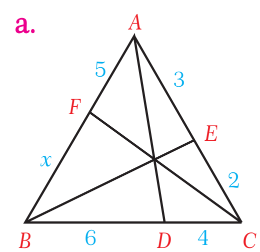




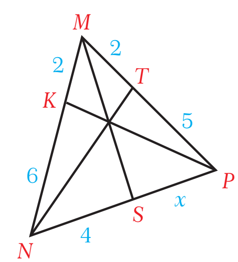
(Hint: Draw a line parallel to DE through B.)

24. Find the length x in each figure by using Ceva's Theorem.

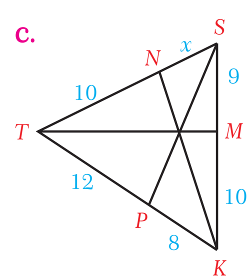
a.



b.



c.



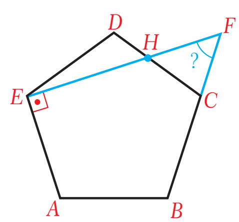
1. What is the sum of the measures of the interior angles of a polygon which has 20 diagonals?

A) 720° B) 900° C) 1080° D) 1800° E) 2160°

2. The measure of an interior angle of a regular polygon is equal to four times the measure of an exterior angle. How many sides does this polygon have?

A) 17 B) 15 C) 12 D) 10 E) 8

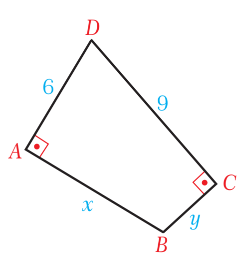
3. In the figure, $ABCDE$ is a regular polygon, points B, C and F are collinear and $FE \perp AE$. What is the measure of $\angle HFC$?



A) 60° B) 54° C) 48° D) 40° E) 8

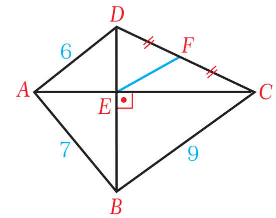
4. In a quadrilateral $ABCD$, $m(\angle A) = m(\angle C) = 90°$, $AD = 6$, $CD = 9$, $AB = x$ and $BC = y$.

If $x + y = 9$, what is the value of $x - y$?



A) 5 B) 6 C) 7 D) 8 E) 9

5. In the quadrilateral $ABCD$ in the figure, diagonals AC and BD are perpendicular to each other, $AB = 7$ cm, $BC = 9$ cm, $AD = 6$ cm and $DF = FC$. Find the length of EF.

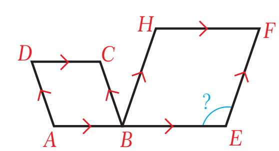


A) $\sqrt{15}$ cm B) $\sqrt{17}$ cm C) $\sqrt{19}$ cm D) $\sqrt{23}$ cm E) $2\sqrt{17}$ cm

6. The diagonals of a quadrilateral $ABCD$ are perpendicular to each other. Which shape is formed by joining the midpoints of the sides of this quadrilateral?

A) a square B) a trapezoid C) a kite D) a parallelogram E) a rectangle

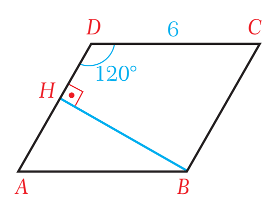
7. In the figure, $ABCD$ and $BEFH$ are parallelograms and points A, B and E are collinear.



If $m(\angle DCB) = 110°$ and $m(\angle CBH) = 30°$, what is $m(\angle BEF)$?

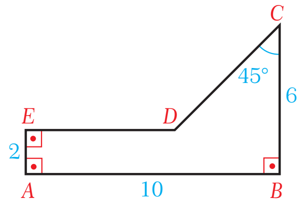
A) 110° B) 100° C) 90° D) 95° E) 80°

8. In the figure, $ABCD$ is a parallelogram with $HB \perp AD$, $m(\angle D) = 120°$ and $DC = 6$ cm. Find the length of BH.



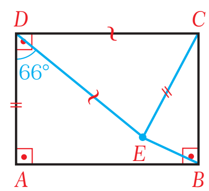
A) 3 cm B) 4 cm C) $3\sqrt{3}$ cm D) $4\sqrt{3}$ cm E) 6 cm

9. In the polygon $ABCDE$ at the right, $\angle A$, $\angle B$ and $\angle E$ are right angles and $m(\angle C) = 45°$. If $AE = 2$, $AB = 10$ and $BC = 6$, what is the perimeter of $ABCDE$?



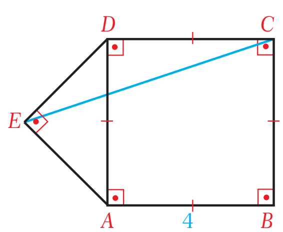
A) $24 + 4\sqrt{2}$ B) $28 + 4\sqrt{2}$ C) $28\sqrt{2}$ D) $30 + 2\sqrt{2}$ E) $4\sqrt{2} + 32$

10. In the figure, $ABCD$ is a rectangle. Given $AD = CE$, $DE = DC$ and $m(\angle ADE) = 66°$, find $m(\angle EBA)$.



A) 5° B) 6° C) 9° D) 10° E) 12°

11. In the figure, $ABCD$ is a square and ΔAED is an isosceles right triangle. If $AB = 4$ cm, what is the length of EC?

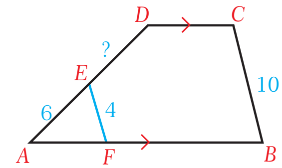


A) $5\sqrt{15}$ cm B) 12 cm C) 10 cm D) $2\sqrt{10}$ cm E) $3\sqrt{8}$ cm

12. The diagonals of a rhombus measure 20 cm and 48 cm respectively. What is its perimeter?

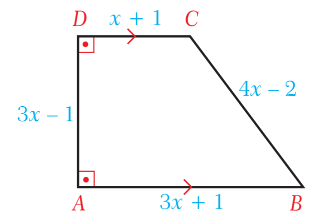
A) 52 cm B) 78 cm C) 104 cm D) 125 cm E) 208 cm

13. In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$. $EF \parallel BC$, $EF = 4$ cm, $BC = 10$ cm and $EA = 6$ cm are given. What is the length of ED?



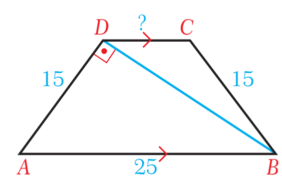
A) 9 cm B) 12 cm C) 15 cm D) 16 cm E) 18 cm

14. In the figure, $ABCD$ is a right trapezoid with $AB \parallel DC$. $AB = 3x + 1$, $BC = 4x - 2$, $AD = 3x - 1$ and $DC = x + 1$ are given. What is the perimeter of this trapezoid?



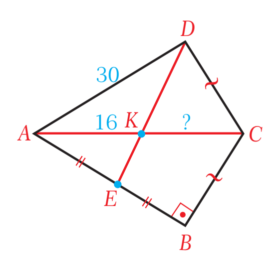
A) 16 B) 20 C) 27 D) 30 E) 32

15. In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$, $AD \perp BD$, $AB = 25$ cm and $AD = BC = 15$ cm. What is the length of DC?



A) 3 cm B) 5 cm C) 6 cm D) 7 cm E) 10 cm

16. In the figure, $ABCD$ is a kite and $AB = AD$. Given that $AB \perp BC$, $AE = EB$, $AK = 16$ and $AD = 30$, find the length of KC.



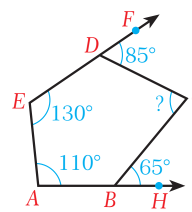
A) $\dfrac{27}{2}$ B) $\dfrac{43}{2}$ C) 18 D) 21 E) 32

1. Three interior angles of a polygon measure 80°, 115° and 135°, and all the other interior angles measure 165°. How many sides does this polygon have?

A) 9 B) 10 C) 12 D) 13 E) 15

2. In the figure, $ABCDE$ is a pentagon.

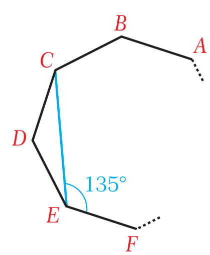
Given that $m(\angle A) = 110°$, $m(\angle E) = 130°$, $m(\angle CBH) = 65°$ and $m(\angle FDC) = 85°$, find $m(\angle BCD)$.



A) 70° B) 75° C) 80° D) 85° E) 90°

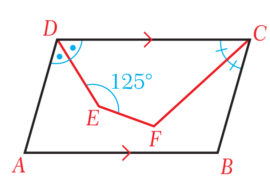
3. In the figure, $ABCDEF$... is a regular polygon.

If $m(\angle CEF) = 135°$, what is the measure of one interior angle of the polygon?



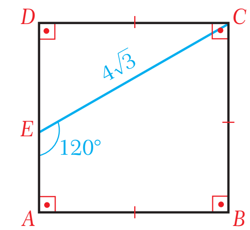
A) 150° B) 145° C) 140° D) 135° E) 130°

4. In the figure, $ABCD$ is a parallelogram and DE and CF are the bisectors of $\angle D$ and $\angle C$ respectively. If $m(\angle DEF) = 125°$, what is $m(\angle EFC)$?



A) 135° B) 140° C) 145° D) 150° E) 155°

5. In the figure, $ABCD$ is a square with $m(\angle CEA) = 120°$ and $CE = 4\sqrt{3}$ cm. Find the perimeter of the square.



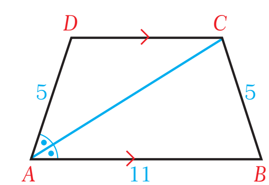
A) 18 cm B) 24 cm C) 30 cm D) 36 cm E) 48 cm

6. The ratio of the lengths of two consecutive sides of a rectangle is 3 : 4. If the perimeter of the rectangle is 42 cm, how long is its diagonal?

A) 9 cm B) 12 cm C) 13 cm D) 14 cm E) 15 cm

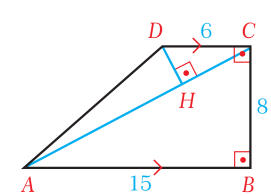
7. In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$, and diagonal AC is the bisector of $\angle A$.

If $AB = 11$ cm and $AD = BC = 5$ cm, what is the length of AC?



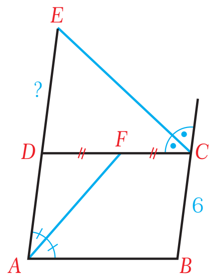
A) 6 cm B) 8 cm C) $3\sqrt{5}$ cm D) $4\sqrt{5}$ cm E) $6\sqrt{5}$ cm

8. In the figure, $ABCD$ is a right trapezoid with $AB \parallel DC$, $DH \perp AC$, $AB = 15$, $BC = 8$ and $DC = 6$. How long is DH?



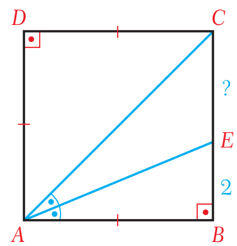
A) $\frac{48}{17}$ B) $\frac{36}{17}$ C) $\frac{32}{17}$ D) $\frac{24}{17}$ E) $\frac{20}{17}$

9. In the figure, $ABCD$ is a parallelogram and points A, D and E are collinear. CE and AF bisect $\angle C'$ and $\angle A$ respectively. If $DF = FC$ and $BC = 6$ cm, what is the length of ED?



A) 14 cm B) 12 cm C) 8 cm D) $4\sqrt{3}$ cm E) 6 cm

10. In the figure, $ABCD$ is a square and AE is the bisector of $\angle CAB$. If $EB = 2$ cm, what is the length of CE?

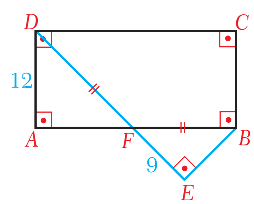


A) $\sqrt{2} - 1$ cm B) $\sqrt{2}$ cm C) $\sqrt{2} + 1$ cm D) $2\sqrt{2}$ cm E) $2\sqrt{2} - 1$ cm

11. In the figure, $ABCD$ is a rectangle.

Given that $DE \perp EB$,

$DF = FB$, $AD = 12$ cm and $FE = 9$ cm, find the length of DC.



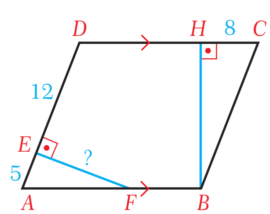
A) 15 cm B) 22 cm C) 24 cm D) 25 cm E) 28 cm

12. In the figure, $ABCD$ is a parallelogram.

Given that $EF \perp AD$,

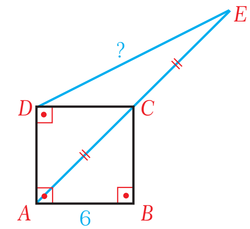
$BH \perp DC$, $AE = 5$,

$DE = 12$ and $HC = 8$, find the length of EF.



A) $\frac{75}{8}$ B) $\frac{25}{3}$ C) 16 D) $\frac{56}{5}$ E) 28

13. In the figure, $ABCD$ is a square and points A, C and E are collinear. If $AC = CE$ and $AB = 6$ cm, how long is ED?

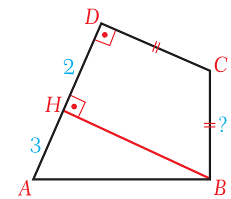


A) 12 cm B) 10 cm C) $6\sqrt{5}$ cm D) $4\sqrt{5}$ cm E) $5\sqrt{2}$ cm

14. In the figure, $ABCD$ is a kite with $AB = AD$ and $DC = CB$.

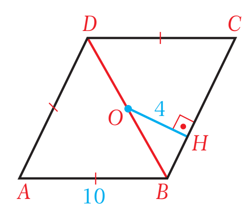
If $HB \perp AD$, $AD \perp DC$,

$AH = 3$ and $HD = 2$, how long is BC?



A) 5 B) $\frac{5}{2}$ C) 32 D) 12 E) $\frac{7}{2}$

15. In the figure, $ABCD$ is a rhombus, point O is the midpoint of the diagonal BD and $OH \perp BC$. If $AB = 10$ cm and $OH = 4$ cm, what is the length of BD?



A) $2\sqrt{5}$ cm B) 5 cm C) 6 cm D) 8 cm E) $4\sqrt{5}$ cm

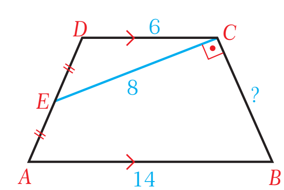
16. In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$.

Given that $AE = ED$,

$EC \perp CB$, $DC = 6$ cm,

$EC = 8$ cm and

$AB = 14$ cm, find the length of BC.

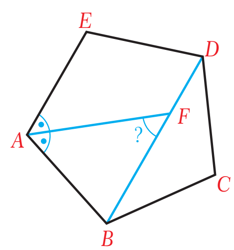


A) 9 cm B) 12 cm C) 15 cm D) 16 cm E) 18 cm

1. The difference between the measures of an interior angle and an exterior angle of a regular polygon is 132°. What is the measure of one interior angle of this polygon?

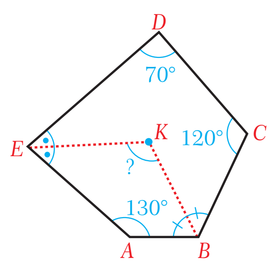
A) 108° B) 120° C) 140° D) 144° E) 156°

2. In the figure, $ABCDE$ is a regular pentagon, DB is a diagonal and AF is the bisector of $\angle A$. What is $m(\angle AFB)$?



A) 54° B) 56° C) 60° D) 72° E) 76°

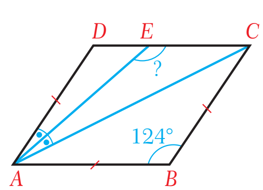
3. In polygon $ABCDE$, EK and BK are the bisectors of $\angle E$ and $\angle B$ respectively. If $m(\angle A) = 130°$, $m(\angle C) = 120°$ and $m(\angle D) = 70°$, what is the measure of $\angle EKB$?



A) 125° B) 120° C) 110° D) 105° E) 100°

4. In the figure, $ABCD$ is a rhombus, AC is its diagonal and AE is the bisector of $\angle DAC$.

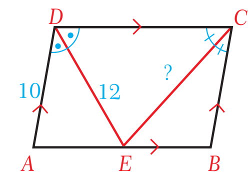
If $m(\angle B) = 124°$, what is $m(\angle AEC)$?



A) 124° B) 130° C) 138° D) 143° E) 146°

5. In the figure, $ABCD$ is a parallelogram and DE and CE are the bisectors of $\angle D$ and $\angle C$ respectively.

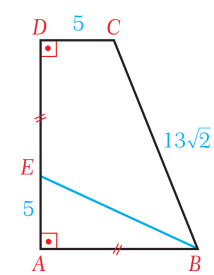
If $AD = 10$ and $DE = 12$, how long is EC?



A) 9 B) 12 C) 16 D) 18 E) 20

6. In the figure, $ABCD$ is a right trapezoid with $AB \parallel DC$.

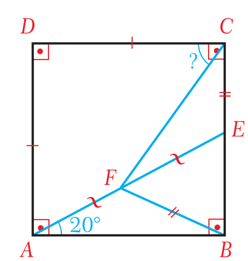
If $AE = DC = 5$, $AB = DE$ and $BC = 13\sqrt{2}$, what is the length of AB?



A) 18 B) 15 C) 13 D) 12 E) 9

7. In the figure, $ABCD$ is a square. $CE = BF$, $AF = FE$ and $m(\angle EAB) = 20°$ are given.

What is the measure of $\angle DCF$?



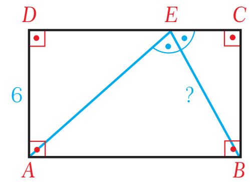
A) 20° B) 30° C) 45° D) 50° E) 55°

8. The radius of the circumscribed circle of a square is 5. What is the radius of the inscribed circle of this square?

A) $2\sqrt{5}$ B) $\dfrac{5\sqrt{2}}{2}$ C) $5\sqrt{2}$ D) 10 E) 12

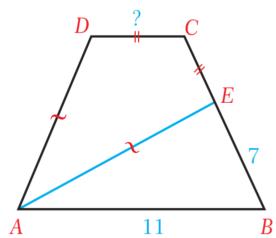
9. In the figure, $ABCD$ is a rectangle and EB bisects $\angle AEC$.

If $DC = 10$ and $AD = 6$, what is the length of BE?



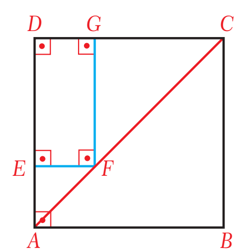
A) $\sqrt{10}$ B) $2\sqrt{10}$ C) $3\sqrt{10}$ D) $\dfrac{3\sqrt{10}}{2}$ E) 7

10. In the figure, $ABCD$ is a trapezoid and $ADCE$ is a kite. Given that $AB \parallel DC$, $DC = CE$, $EB = 7$ and $AB = 11$, find the length of CD.



A) 5 B) $\sqrt{5}$ C) 4 D) 3 E) $\sqrt{2}$

11. In the figure, $ABCD$ is a square and $DEFG$ is a rectangle. If $DE = 2 \cdot EF$ and $AC = 12\sqrt{2}$, what is the perimeter of the rectangle $DEFG$?



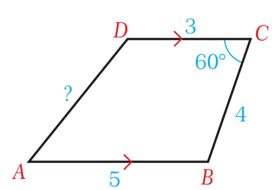
A) 12 B) 18 C) 20 D) 24 E) 32

12. In the figure, $ABCD$ is a trapezoid with $AB \parallel DC$.

If $m(\angle C) = 60°$,

$AB = 5$, $BC = 4$

and $DC = 3$, what is the length of AD?

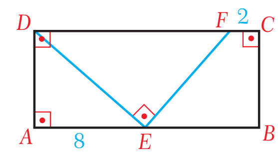


A) 5 B) $2\sqrt{7}$ C) 6 D) $2\sqrt{10}$ E) 7

13. In the figure, $ABCD$ is a rectangle.

Given $DE \perp EF$,

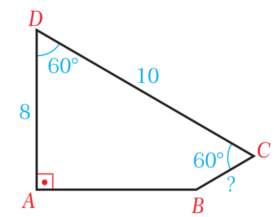
$DE = 2 \cdot EF$, $AE = 8$ cm and $FC = 2$ cm, find the length of BC.



A) 4 cm B) 5 cm C) 6 cm D) 8 cm E) 10 cm

14. In a quadrilateral $ABCD$, $AB \perp AD$ and $m(\angle C) = m(\angle D) = 60°$.

If $AD = 8$ cm and $DC = 10$ cm, what is the length of BC?



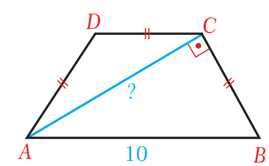
A) $2\sqrt{3}$ cm B) $3\sqrt{2}$ cm C) 6 cm D) 5 cm E) $3\sqrt{3}$ cm

15. In the figure, $ABCD$ is an isosceles trapezoid with $AB \parallel DC$.

If $AD = DC = CB$,

$m(\angle ACB) = 90°$ and

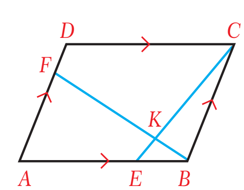
$AB = 10$, what is the length of AC?



A) $2\sqrt{3}$ B) $3\sqrt{2}$ C) $4\sqrt{2}$ D) $5\sqrt{3}$ E) $6\sqrt{2}$

16. In the figure, $ABCD$ is a parallelogram.

If $AE = 2 \cdot EB$, $AF = 3 \cdot FD$ and $KB = 4$, what is the length of FK?



A) 9 B) 11 C) 15 D) 16 E) 18

PYTHAGOREAN THEOREM

a. The Pythagorean Theorem

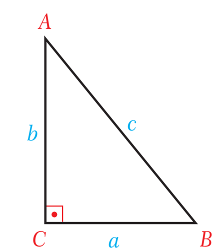
The Pythagorean Theorem is one of the most famous theorems in Euclidean geometry, and almost everyone with a high school education can remember it.

Theorem

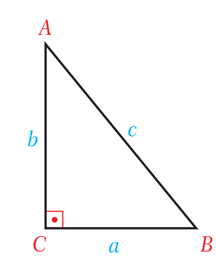
Pythagorean Theorem

In a right triangle ABC with $m(\angle C) = 90°$, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs, i.e.

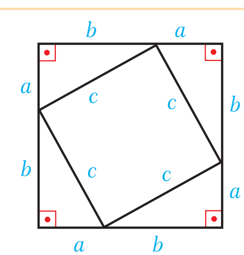
$$c^2 = b^2 + a^2.$$



Proof There are many proofs of the Pythagorean Theorem. The proof we will give here uses the dissection of a square. It proves the Pythagorean Theorem for the right triangle ABC shown opposite.



Imagine that a large square with side length $a + b$ is dissected into four congruent right triangles and a smaller square, as shown in the figure. The legs of the triangles are a and b, and their hypotenuse is c. So the smaller square has side length c.



We can now write two expressions for the area S of the larger square:

$$S = 4 \cdot \left(\frac{a \cdot b}{2}\right) + c^2 \quad \text{and } S = (a + b)^2.$$

Since these expressions are equal, we can write

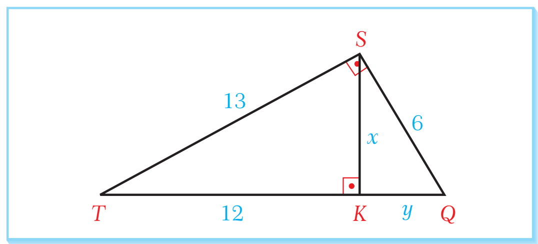
$$4 \cdot \left(\frac{a \cdot b}{2}\right) + c^2 = (a + b)^2$$

$$2ab + c^2 = a^2 + 2ab + b^2$$

$$c^2 = a^2 + b^2.$$

This concludes the proof of the Pythagorean Theorem.

EXAMPLE 1 In the figure, $ST \perp SQ$. Find x and y.



Solution First we will use the Pythagorean Theorem in ΔSKT to find x, then we can use it in ΔSKQ to find y.

- $SK^2 + KT^2 = ST^2$ (Pythagorean Theorem in ΔSKT)

  $x^2 + 12^2 = 13^2$ (Substitute)

  $x^2 + 144 = 169$

  $x^2 = 25$ (Simplify)

  $x = 5$ (Positive length)

- $SK^2 + KQ^2 = SQ^2$ (Pythagorean Theorem in ΔSKQ)

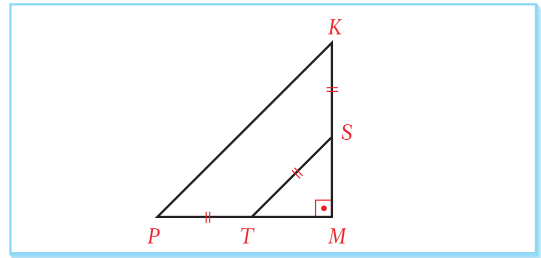
  $5^2 + y^2 = 6^2$ (Substitute)

  $y^2 = 36 - 25$ (Simplify)

  $y = \sqrt{11}$

$x = -5$ is not an answer because the length of a segment cannot be negative. So the answer is $x = 5$. From now on we will always consider only positive values for lengths.

EXAMPLE 2 In the figure,
$PT = TS = KS$,
$PM = 4$ cm and $KM = 3$ cm. Find ST.



Solution Let $PT = TS = KS = x$.

So $SM = KM - KS = 3 - x$ and $TM = PM - PT = 4 - x$.

Quadratic formula
The roots x_1 and x_2 of the quadratic equation $ax^2 + bx + c = 0$ are $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

In ΔTMS, $TS^2 = TM^2 + MS^2$ (Pythagorean Theorem)

$x^2 = (4 - x)^2 + (3 - x)^2$ (Substitute)

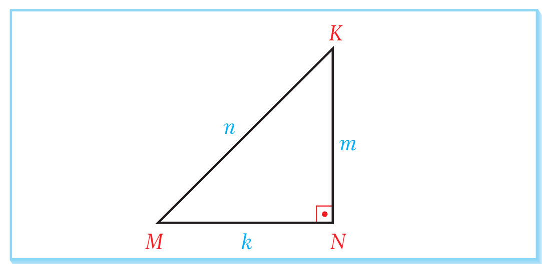
$x^2 = 16 - 8x + x^2 + 9 - 6x + x^2$ (Simplify)

$x^2 - 14x + 25 = 0$

$x_{1,2} = (7 \pm \sqrt{24})$ cm (Quadratic formula)

Since $7 + \sqrt{24}$ is greater than 3 and 4 which are the lengths of the sides, the answer is $x = |ST| = 7 - \sqrt{24}$ cm.

EXAMPLE 3 In the figure,
$m + k = 3 \cdot n$.
Given $A(\Delta KMN) = 30\ \text{cm}^2$,
find the value of n.



Solution

- $m + k = 3 \cdot n$ (1) (Given)
- $A(\Delta KMN) = 30\ \text{cm}^2$ (Given)

  $\dfrac{k \cdot m}{2} = 30$ (Definition of the area of a triangle)

  $k \cdot m = 60$ (2)
- In ΔKMN, $n^2 = k^2 + m^2$ (Pythagorean Theorem)

  $n^2 = (k + m)^2 - 2km$ (Binomial expansion: $(k + m)^2 = k^2 + 2km + m^2$)

  $n^2 = (3n)^2 - 2 \cdot 60$ (Substitute (1) and (2))

  $8n^2 = 120$ (Simplify)

  $n^2 = 15$

  $n = \sqrt{15}$ cm.

Theorem

Converse of the Pythagorean Theorem

If the square of one side of a triangle equals the sum of the squares of the other two sides, then the angle opposite this side is a right angle.

Proof We will give a proof by contradiction.

Suppose the triangle is not a right triangle, and label the vertices A, B and C. Then there are two possibilities for the measure of angle C: either it is less than 90° (figure 1), or it is greater than 90° (figure 2).

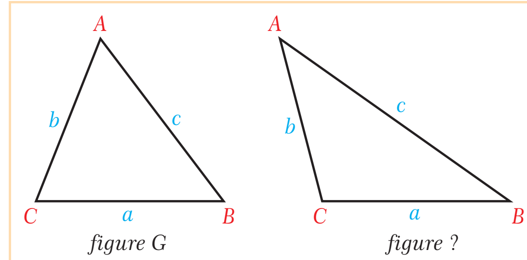


figure G figure ?

Let us draw a segment $DC \perp CB$ such that $DC = AC$.

By the Pythagorean Theorem in ΔBCD, $BD^2 = a^2 + b^2 = c^2$, and so $BD = c$.

So ΔACD is isosceles (since $DC = AC$) and ΔABD is also isosceles ($AB = BD = c$). As a result, $\angle CDA \cong \angle CAD$ and $\angle BDA \cong \angle DAB$.

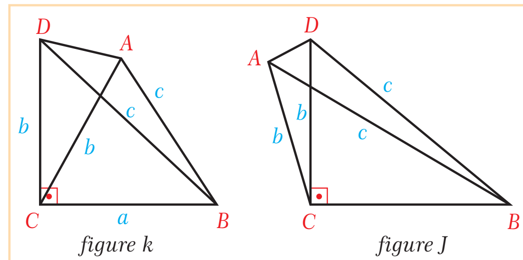
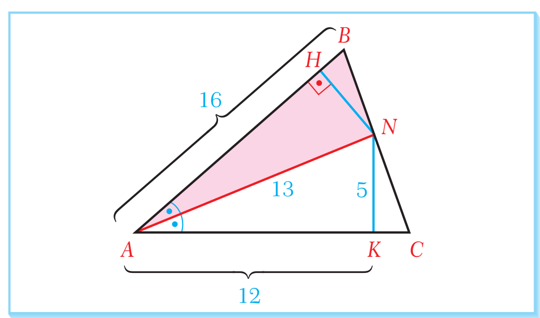


figure k figure J

However, in figure 3 we have $m(\angle BDA) < m(\angle CDA)$ and $m(\angle CAD) < m(\angle DAB)$, which gives $m(\angle BDA) < m(\angle DAB)$ if $\angle CDA$ and $\angle CAD$ are congruent. This is a contradiction of $\angle BDA \cong \angle DAB$. Also, in figure 4 we have $m(\angle DAB) < m(\angle CAD)$ and $m(\angle CDA) < m(\angle BDA)$, which gives $m(\angle DAB) < m(\angle BDA)$ if $\angle CAD$ and $\angle CDA$ are congruent. This is also a contradiction.

So our original assumption must be wrong, and so ΔABC is a right triangle.

EXAMPLE 4 In the triangle ABC opposite, $K \in AC$ and AN is the interior angle bisector of $\angle A$. $AB = 16$ cm, $AN = 13$ cm, $AK = 12$ cm and $NK = 5$ cm are given. Find the area of ΔABN.



Solution Let us draw an altitude NH from the vertex N to the side AB.

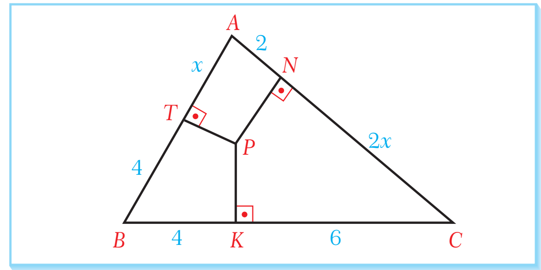
To find the area of ΔABN we need to find NH, because $A(\Delta ABN) = \dfrac{NH \cdot AB}{2}$ and AB is given as 16 cm.

$13^2 = 12^2 + 5^2$, so $m(\angle NKA) = 90°$. (Converse of the Pythagorean Theorem)

Also, $NH = NK = 5$ cm. (Angle Bisector Theorem)

So $A(\Delta ABN) = \dfrac{NH \cdot AB}{2} = \dfrac{5 \cdot 16}{2} = 40\ \text{cm}^2$. (Substitution)

EXAMPLE 5 Find the length x in the figure.



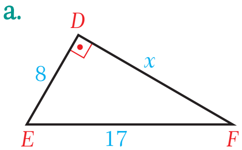
Solution

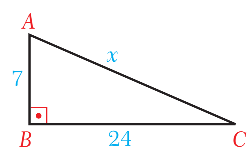
$$AT^2 + BK^2 + CN^2 = AN^2 + BT^2 + CK^2 \quad \text{(Carnot's Theorem)}$$
$$x^2 + 4^2 + (2x)^2 = 2^2 + 4^2 + 6^2 \quad \text{(Substitute)}$$
$$5x^2 = 40 \quad \text{(Simplify)}$$
$$x^2 = 8$$
$$x = 2\sqrt{2}$$

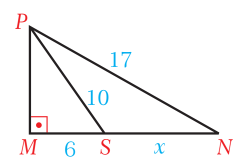
A **Pythagorean triple** is a set of three integers a, b and c which satisfy the Pythagorean Theorem. The smallest and best-known Pythagorean triple is $(a, b, c) = (3, 4, 5)$.

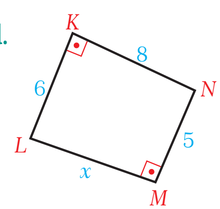
Check Yourself

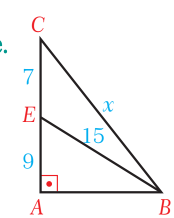
1. Find the length x in each figure.

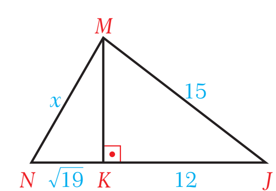
a.


b.


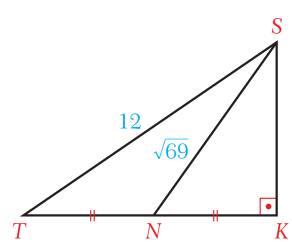
c.


d.


e.


f.


2. In the figure, $TN = NK$, $ST = 12$ cm and $SN = \sqrt{69}$ cm. Find the length of TK.



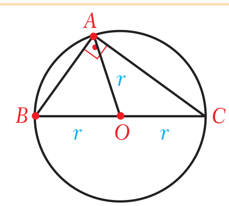
3. In a right triangle ABC, $m(\angle A) = 90°$, $AB = x$, $AC = x - 7$ and $BC = x + 1$. Find AC.

Answers

1. a. 15 b. 25 c. 9 d. $5\sqrt{3}$ e. 20 f. 10 2. 10 cm 3. 5 cm

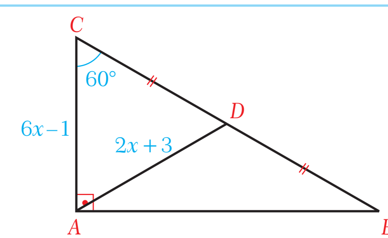
Properties 7

1. The length of the median to the hypotenuse of a right triangle is equal to half of the length of the hypotenuse.
2. a. In any isosceles right triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of a leg. (This property is also called the **45°-45°-90° Triangle Theorem**.)
   b. In any right triangle, if the hypotenuse is $\sqrt{2}$ times any of the legs then the triangle is a 45°-45°-90° triangle. (This property is also called the **Converse of the 45°-45°-90° Triangle Theorem)**.
3. In any 30°-60°-90° right triangle,
   a. the length of the hypotenuse is twice the length of the leg opposite the 30° angle.
   b. the length of the leg opposite the 60° angle is $\sqrt{3}$ times the length of the leg opposite the 30° angle. (These properties are also called the **30°-60°-90° Triangle Theorem**.)
4. In any right triangle,
   a. if one of the legs is half the length of the hypotenuse then the angle opposite this leg is 30°.
   b. if one of the legs is $\sqrt{3}$ times the length of the other leg then the angle opposite this first leg is 60°. (These properties are also called the **Converse of the 30°-60°-90° Triangle Theorem.**)
5. The center of the circumscribed circle of any right triangle is the midpoint of the hypotenuse of the triangle.



EXAMPLE 6

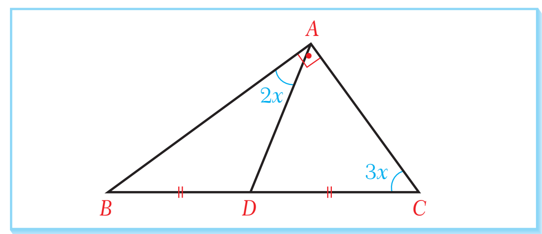
In the figure,
$m(\angle BAC) = 90°$,
$m(\angle C) = 60°$ and
$BD = DC$.
Find BC if $AD = 2x + 3$ and $AC = 6x - 1$.



Solution

- Since AD is a median and the length of the median to the hypotenuse of a right triangle is equal to half the length of the hypotenuse, $AD = \frac{1}{2} \cdot BC$.
- By the Triangle Angle-Sum Theorem in ΔABC, $m(\angle B) = 30°$.
- By the 30°-60°-90° Triangle Theorem, $AC = \frac{1}{2} \cdot BC$ because $m(\angle B) = 30°$ and BC is the hypotenuse.
- So by the transitive property of equality, $AC = AD$, i.e. $6x - 1 = 2x + 3$ and so $x = 1$.
- Finally, $BC = 2 \cdot AC = 2 \cdot AD = 2 \cdot (2x + 3) = 10$.

EXAMPLE 7 In the figure at the right, find $m(\angle ADC)$ if
$m(\angle BAC) = 90°$,
$m(\angle BAD) = 2x$,
$m(\angle ACB) = 3x$ and
$BD = DC$.



Solution Since AD is a median, by Property 7.1 we have $AD = \frac{1}{2} \cdot BC$.

So $AD = BD = DC$. Hence ΔDCA and ΔBDA are isosceles triangles.

Since ΔDCA is isosceles, $m(\angle DAC) = m(\angle ACD) = 3x$.

Additionally, $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$ by the Angle Addition Postulate.

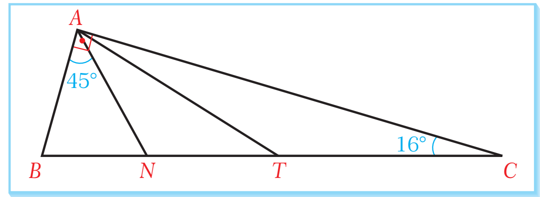
So $2x + 3x = 90°$ and $x = 18°$.

By the Triangle Angle-Sum Theorem in ΔDCA, $m(\angle ADC) + 3x + 3x = 180°$.

So $m(\angle ADC) = 180° - (6 \cdot 18)°$, i.e. $m(\angle ADC) = 72°$.

EXAMPLE 8 One of the acute angles in a right triangle measures 16°. Find the angle between the bisector of the right angle and the median drawn from the same vertex.

Solution Let us draw an appropriate figure. We need to find $m(\angle NAT)$.



According to the figure,

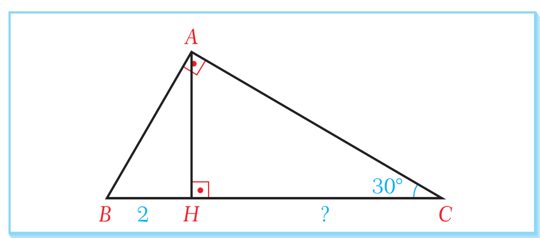
- AN is the angle bisector, AT is the median, and $m(\angle BAC) = 90°$.
- $m(\angle ACB) = 16°$ by Property 5.3.
- Since AT is median to hypotenuse, $AT = CT = BT$.
- So ΔATC is isosceles.
- Therefore, by the Isosceles Triangle Theorem, $m(\angle TAC) = m(\angle ACT) = 16°$.
- Since AN is an angle bisector and $m(\angle BAC) = 90°$, $m(\angle NAC) = 45°$.
- So $m(\angle NAT) = m(\angle NAC) - m(\angle TAC) = 45° - 16° = 29°$.

Property 5.3:

In any triangle ABC, if $m(\angle B) > m(\angle C)$ or $m(\angle B) < m(\angle C)$ then $h_a < n_a < V_a$.

EXAMPLE 9 In the figure, $AB \perp AC$ and $AH \perp BC$.

Given $m(\angle C) = 30°$ and $BH = 2$ cm, find the length of HC.



Solution In ΔABC, since $m(\angle C) = 30°$,

$m(\angle B) = 60°$.

In ΔABH, since $m(\angle B) = 60°$,

$m(\angle BAH) = 30°$.

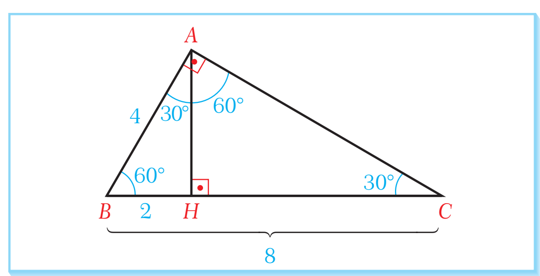
In ΔABH, by Property 7.3,

$AB = 2 \cdot BH = 2 \cdot 2 = 4$ cm.

In ΔABC, again by Property 7.3,

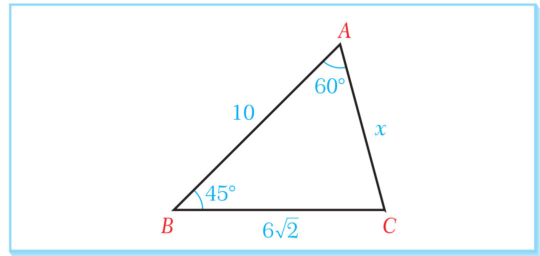
$BC = 2 \cdot AB = 2 \cdot 4 = 8$ cm.

So $HC = BC - BH = 8 - 2 = 6$ cm.



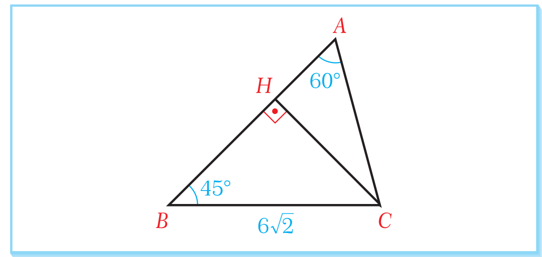
This set square is in the form of a 30°-60°-90° triangle.

EXAMPLE 10 Find the value of x in the figure.



Solution Let us draw an altitude from C to AB.

This set square is in the form of 45°-45°-90° right triangle.



- In ΔBHC,

$BC = \sqrt{2} \cdot BH$	(45°-45°-90° Triangle Theorem)
$6\sqrt{2} = \sqrt{2} \cdot BH$	(Substitute)
$BH = 6.$	(Simplify)

- $AB = AH + HB$ (Segment Addition Postulate)

$10 = AH + 6$	(Substitute)
$AH = 4.$	(Simplify)

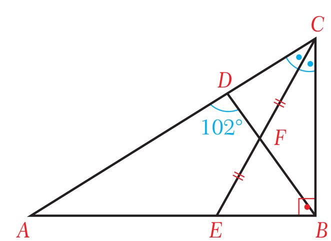
- In ΔAHC,

$AC = 2 \cdot AH$	(30°-60°-90° Triangle Theorem)
$AC = 2 \cdot 4$	(Substitute)
$AC = x = 8.$	(Simplify)

Objectives

1. In an isosceles right triangle, the sum of the lengths of the hypotenuse and the altitude drawn to the hypotenuse is 27.3. Find the length of the hypotenuse.

2. In the figure, ΔABC is a right triangle with $m(\angle ABC) = 90°$ and $CF = FE$, and CE is the angle bisector of $\angle C$. If $m(\angle ADB) = 102°$, find the measure of $\angle CAB$.

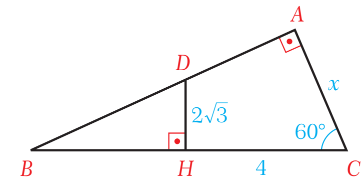


3. One of the acute angles in a right triangle measures 48°. Find the angle between the median and the altitude which are drawn from the vertex at the right angle.

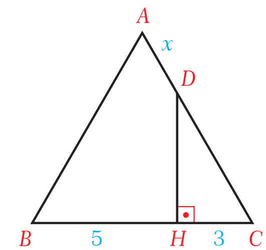
4. In a triangle ABC, $m(\angle B) = 135°$, $AC = 17$ cm and $BC = 8\sqrt{2}$ cm. Find the length of AB.

5. In a right triangle, the sum of the lengths of the hypotenuse and the shorter leg is 2.4. Find the length of the hypotenuse if the biggest acute angle measures 60°.

6. In the figure,
$m(\angle C) = 60°$,
$HC = 4$ cm and
$DH = 2\sqrt{3}$ cm. Find the length $AC = x$.



7. ΔABC in the figure is an equilateral triangle with
$DH \perp BC$,
$BH = 5$ cm and
$HC = 3$ cm.
Find the length $AD = x$.



8. The distance from a point to a line k is 10 cm. Two segments non-perpendicular to k are drawn from this point. Their lengths have the ratio $2 : 3$. Find the length of the longer segment if the shorter segment makes a 30° angle with k.
9. ΔCAB is a right triangle with $m(\angle A) = 90°$ and $m(\angle C) = 60°$, and D is the midpoint of hypotenuse. Find the length of the hypotenuse if $AD = 3x + 1$ and $AC = 5x - 3$.
10. The hypotenuse of an isosceles right triangle measures 18 cm. Find the distance from the vertex at the right angle to the hypotenuse.

Answers

1. 18.2 2. 22° 3. 6° 4. 7 cm 5. 1.6 6. 5 cm 7. 2 cm 8. 30 cm 9. 14 10. 9 cm

The word **trigonometry** is derived from the Greek words *trigon* (which means ‘triangle’) and *metry* (which means ‘measurement’). So **trigonometry** is the study of triangle measurement. In this chapter we will study trigonometry for right triangles.

Remember that a right triangle is a triangle with one 90° angle and two acute angles. Let us begin by looking at the basics of trigonometry in a right triangle.

A. TRIGONOMETRIC RATIOS OF ACUTE ANGLES

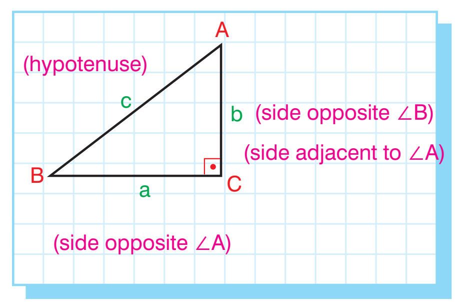
A trigonometric ratio is the ratio of the lengths of any two sides of a right triangle. We will learn three basic trigonometric ratios: sine, cosine, and tangent. We abbreviate them as sin, cos, and tan respectively.

Let DABC be a right triangle. Then the basic trigonometric ratios are defined as follows:

$$\sin\angle A = \frac{\text{length of the side opposite } \angle A}{\text{length of hypotenuse}}$$

$$\cos\angle A = \frac{\text{length of the side adjacent to } \angle A}{\text{length of hypotenuse}}$$

$$\tan\angle A = \frac{\text{length of the side opposite } \angle A}{\text{length of the side adjacent to } \angle A}$$



Therefore, in the right triangle above,

$$\sin\angle A=\frac{a}{c},\ \cos\angle A=\frac{b}{c},\ \tan\angle A=\frac{a}{b}, \text{ and } \sin\angle B=\frac{b}{c},\ \cos\angle B=\frac{a}{c},\ \tan\angle B=\frac{b}{a}.$$

Note

We sometimes use tg as the abbreviation of tangent.

Activity Time

1. Draw at least four non-congruent right triangles containing an angle of 30°.
2. Make a table with six columns and as many rows as the number of triangles.
3. Measure the length of each side of each triangle in millimeters. Write the lengths in the first three columns of the table.
4. Use a calculator to find the following values for each triangle:

   a. $\dfrac{\text{length of the side opposite } 30°}{\text{length of hypotenuse}}$.

b. $\dfrac{\text{length of the side adjacent to } 30^\circ}{\text{length of hypotenuse}}$.

c. $\dfrac{\text{length of the side opposite } 30^\circ}{\text{length of the side adjacent to } 30^\circ}$.

Write the results in the last three columns of your table.

5. What can you say about the numbers in the last three columns of your table?

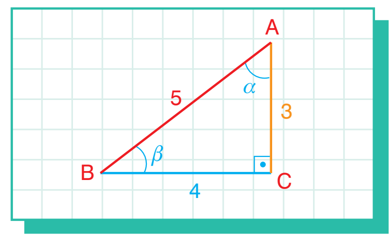
6. Repeat steps 1-5 for triangles containing an angle of 53°.

EXAMPLE 11 Find the trigonometric ratios in the given triangle.

a. sin a

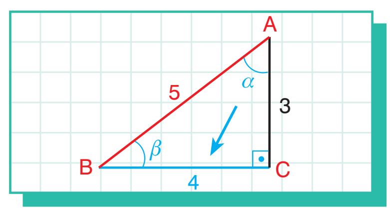
b. cos a

c. tan a

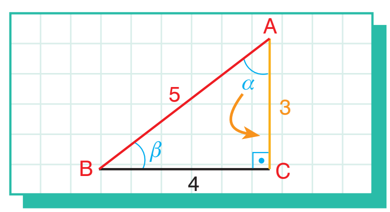


Solution

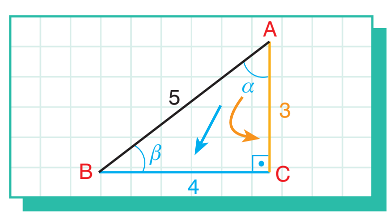
a. $\sin\alpha = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{4}{5}$



b. $\cos\alpha = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{3}{5}$

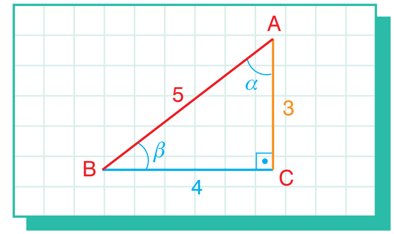


c. $\tan\alpha = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{4}{3}$



EXAMPLE 12 Find the trigonometric ratios in the triangle.

a. sin a b. cos a c. tan a

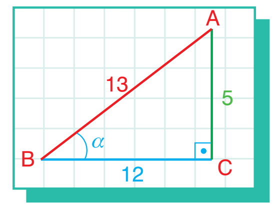


Solution In order to find all the ratios we need to find |BC|.

Remember the Pythagorean Theorem:

$$|AC|^2 + |BC|^2 = |AB|^2$$

$$\left.\begin{array}{l} 5^2 + |BC|^2 = 13^2 \\ |BC|^2 = 169 - 25 \\ \text{So } |BC| = 12 \text{ and} \\ |BC|2 = 144 \\ |BC| = 12 \end{array}\right\}$$



a. $\sin\alpha = \dfrac{5}{13}$.

b. $\cos\alpha = \dfrac{12}{13}$.

c. $\tan\alpha = \dfrac{5}{12}$.

B. RECIPROCAL TRIGONOMETRIC RATIOS

Note

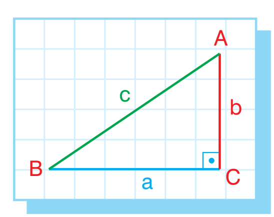
We sometimes use ctg as the abbreviation of cotangent.

We have looked at the basic trigonometric ratios sine, cosine, and tangent. Now we can define three new trigonometric ratios. They are cosecant, secant, and cotangent, which we abbreviate as cosec, sec, and cotan respectively. They are defined as follows:

$$\text{cosec}\,\angle A = \frac{\text{length of hypotenuse}}{\text{length of side opposite }\angle A} = \frac{c}{a} = \frac{1}{\sin\angle A}$$

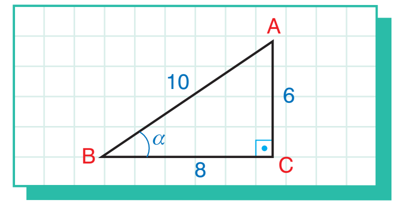
$$\sec\angle A = \frac{\text{length of hypotenuse}}{\text{length of side adjacent to }\angle A} = \frac{c}{b} = \frac{1}{\cos\angle A}$$

$$\cot\angle A = \frac{\text{length of side adjacent to }\angle A}{\text{length of side opposite }\angle A} = \frac{b}{a} = \frac{1}{\tan\angle A}$$



EXAMPLE 13 Write the ratios for the triangle in the figure.

a. cot a b. sec a c. cosec a



Solution

a.
$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{\frac{6}{8}} = 1 \cdot \frac{8}{6} = \frac{8}{6}$$

b.
$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\frac{8}{10}} = 1 \cdot \frac{10}{8} = \frac{10}{8}$$

c.
$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha} = \frac{1}{\frac{6}{10}} = 1 \cdot \frac{10}{6} = \frac{10}{6}$$

C. TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

Since the sum of the acute angles in a right triangle is 90°, these angles are complementary.

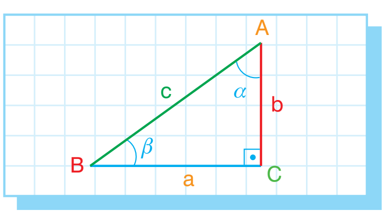
$a + \beta + 90° = 180°$ (sum of the angles in a triangle)

$a + \beta = 180° - 90°$

$a + \beta = 90°$

So $a = 90° - \beta$ or

$\beta = 90° - a$.



In the right triangle opposite,

$$\sin\alpha = \frac{a}{c} = \cos\beta, \quad \cos\alpha = \frac{b}{c} = \sin\beta, \quad \tan\alpha = \frac{a}{b} = \cot\beta, \text{ and}$$

$$\cot\alpha = \frac{b}{a} = \tan\beta, \quad \sec\alpha = \frac{c}{b} = \operatorname{cosec}\beta, \quad \operatorname{cosec}\alpha = \frac{c}{a} = \sec\beta.$$

Therefore,

$$\sin a = \cos b = \cos (90° - a),$$
$$\cos a = \sin b = \sin (90° - a),$$
$$\tan a = \cot b = \cot (90° - a),$$
$$\cot a = \tan b = \tan (90° - a),$$

For example,

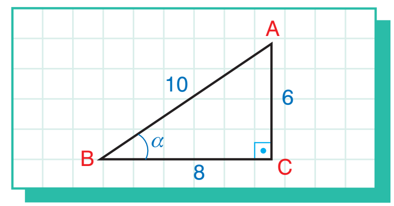
$\sin 53° = \cos (90° - 53°) = \cos 37°$

$\tan 17° = \cot (90° - 17°) = \cot 73°$

$\cos 29° = \sin (90° - 29°) = \cot 61°$.

EXAMPLE 14 Find sin b if

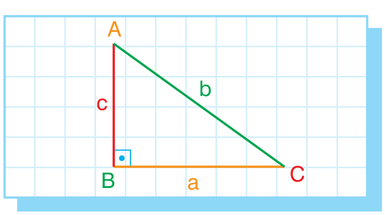
$\cos a = \frac{4}{5}$ and $a + b = 90°$.



Solution Since $a + \beta = 90°$, $\beta = 90° - a$.

We can write $\sin \beta = \sin (90° - a) = \cos a = \frac{4}{5}$

D. BASIC TRIGONOMETRIC IDENTITIES



Look at the right triangle in the figure.

In the triangle, $\sin\angle A = \frac{a}{b}$ and $\sin\angle C = \frac{c}{b}$. So $\tan\angle A = \frac{a}{c}$.

1. If we divide the top and bottom of the fraction for the tangent by b we get

$$\tan\angle A = \frac{\frac{a}{b}}{\frac{c}{b}} = \frac{\sin\angle A}{\cos\angle A} \quad \left(\text{since } \frac{a}{b} = \sin\angle A \text{ and } \frac{c}{b} = \cos\angle A\right).$$

Property

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \quad (\cos\alpha \neq 0)$$

2. If we apply the definition of cotangent we get

$$\cot\angle A = \frac{1}{\tan\angle A} = \frac{1}{\dfrac{\sin\angle A}{\cos\angle A}} = \frac{\cos\angle A}{\sin\angle A}.$$

Property

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} \quad (\sin\alpha \neq 0)$$

3. $\tan a \times \cot a = \cancel{\tan a} \times \overbrace{\frac{\cancel{1}}{\cancel{\tan\alpha}}}^{\cot\alpha} = 1$

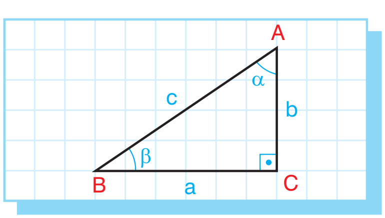
or

$$\tan a \times \cot a = \underbrace{\frac{\cancel{\sin\alpha}}{\cancel{\cos\alpha}}}_{\tan\alpha} \cdot \underbrace{\frac{\cancel{\cos\alpha}}{\cancel{\sin\alpha}}}_{\cot\alpha} = 1.$$

Property

$$\tan a \times \cot a = 1$$

4. Look at the triangle on the left.



We can write $\sin\alpha = \dfrac{a}{c}$ and $\cos\alpha = \dfrac{b}{c}$.

Now $\sin^2 a + \cos^2 a = \left(\dfrac{a}{c}\right)^2 + \left(\dfrac{b}{c}\right)^2 = \underbrace{\dfrac{a^2}{c^2} + \dfrac{b^2}{c^2}} = \dfrac{a^2 + b^2}{c^2}$.

By the Pythagorean Theorem, we know that $a^2 + b^2 = c^2$.

Therefore we have $\sin^2 a + \cos^2 a = \dfrac{(a^2 + b^2)}{c^2} = \dfrac{c^2}{c^2} = 1.$

Conclusion

$\sin^2 \alpha + \cos^2 \alpha = 1$, $\cos^2 \alpha = 1 - \sin^2 \alpha$, and $\sin^2 \alpha = 1 - \cos^2 \alpha$.

We can use the identities we have found to simplify trigonometric expressions.

EXAMPLE 15 Simplify $1 + \tan^2 x$.

Solution

We know $\tan x = \dfrac{\sin x}{\cos x}$.

$$\text{So } 1+\left(\frac{\sin x}{\underset{(\cos^2 x)}{\cos x}}\right)^2 = \frac{1}{1}+\frac{\sin^2 x}{\cos^2 x} = \frac{\overbrace{\cos^2 x+\sin^2 x}^{1}}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 = \sec^2 x.$$

EXAMPLE 16 Simplify $\sin x \times \cot x$.

Solution

$$\cancel{\sin}\,x \cdot \underbrace{\frac{\cos x}{\cancel{\sin}\,x}}_{\cot x} = \cos x$$

EXAMPLE 17 Simplify $\cos x + \tan x \times \sin x$.

Solution

$$\cos x + \underbrace{\frac{\sin x}{\cos x}}_{\tan x}\cdot \sin x = \frac{\cos x}{\underset{(\cos x)}{1}} + \frac{\sin^2 x}{\cos x} = \frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos x} = \frac{\overbrace{\cos^2 x + \sin^2 x}^{1}}{\cos x}$$

$$= \frac{1}{\cos x} = \sec x$$

EXAMPLE 18 Simplify

(cot x – tan x) × (sin x × cos x).

Solution

$$(\underbrace{\frac{\cos x}{\sin x}}_{\cot x} - \underbrace{\frac{\sin x}{\cos x}}_{\tan x}) \cdot \sin x \cdot \cos x = \frac{\cos^2 x - \sin^2 x}{\cancel{\sin x} \cdot \cancel{\cos x}} \cdot \cancel{\sin x} \cdot \cancel{\cos x} = \cos^2 x - \sin^2 x$$

$$= (1 - \sin^2 x) - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

or

$$= \cos^2 x - (\underbrace{1 - \cos^2 x}_{\sin^2 x})$$

$$= \cos^2 x - 1 + \cos^2 x$$

$$= 2\cos^2 x - 1$$

EXAMPLE 19 Simplify $\dfrac{\tan\alpha + \cot\alpha}{\operatorname{cosec}\alpha \cdot \sec\alpha}$.

Solution

$$\frac{\overbrace{\dfrac{\sin\alpha}{\cos\alpha}}^{\tan\alpha} + \overbrace{\dfrac{\cos\alpha}{\sin\alpha}}^{\cot\alpha}}{\underbrace{\dfrac{1}{\sin\alpha}}_{\operatorname{cosec}\alpha} \cdot \underbrace{\dfrac{1}{\cos\alpha}}_{\sec\alpha}} = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\cos\alpha}{\sin\alpha}}{\dfrac{1}{\sin\alpha \cdot \cos\alpha}} = \frac{\dfrac{\overbrace{\sin^2\alpha + \cos^2\alpha}^{1}}{\sin\alpha \cdot \cos\alpha}}{\dfrac{1}{\sin\alpha \cdot \cos\alpha}} = \frac{\dfrac{1}{\sin\alpha \cdot \cos\alpha}}{\dfrac{1}{\sin\alpha \cdot \cos\alpha}} = 1$$

EXAMPLE 20 Simplify $\dfrac{\tan\alpha+\cot\alpha}{\operatorname{cosec}\alpha\cdot\sec\alpha}$.

Solution

$$\frac{\overbrace{\dfrac{\sin\alpha}{\cos\alpha}}^{\tan\alpha}+\overbrace{\dfrac{\cos\alpha}{\sin\alpha}}^{\cot\alpha}}{\underbrace{\dfrac{1}{\sin\alpha}}_{\operatorname{cosec}\alpha}\cdot\underbrace{\dfrac{1}{\cos\alpha}}_{\sec\alpha}}=\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\cos\alpha}{\sin\alpha}}{\dfrac{1}{\sin\alpha\cdot\cos\alpha}}=\frac{\dfrac{\overbrace{\sin^2\alpha+\cos^2\alpha}^{1}}{\sin\alpha\cdot\cos\alpha}}{\dfrac{1}{\sin\alpha\cdot\cos\alpha}}=\frac{\dfrac{1}{\sin\alpha\cdot\cos\alpha}}{\dfrac{1}{\sin\alpha\cdot\cos\alpha}}=1$$

EXAMPLE 21 Simplify $\sin^3 x+\cos^2 x\times\sin x$.

Solution $\underbrace{\sin x\cdot\sin^2 x}_{\sin^3 x}+\cos^2 x\cdot\sin x=\sin x\,(\underbrace{\sin^2 x+\cos^2 x}_{1})=\sin x$

EXAMPLE 22 Simplify

$\cos x\cdot\dfrac{\cot x}{\sec x}\cdot\dfrac{1}{1-\sin^2 x}\cdot\tan x$.

Solution

$$\cos x\cdot\frac{\overbrace{\dfrac{\cos x}{\sin x}}^{\cot x}}{\underbrace{\dfrac{1}{\cos x}}_{\sec x}}\cdot\frac{1}{\underbrace{1-\sin^2 x}_{\cos^2 x}}\cdot\underbrace{\frac{\sin x}{\cos x}}_{\tan x}=\cos x\cdot\frac{\cos x}{\sin x}\cdot\cos x\cdot\frac{1}{\cos^2 x}\cdot\frac{\sin x}{\cos x}=1$$

EXAMPLE 23 Simplify tan x + cot x.

Solution

$$\underbrace{\frac{\sin x}{\cos x}}_{\tan x}+\underbrace{\frac{\cos x}{\sin x}}_{\cot x}=\frac{\sin x}{\underset{(\sin x)}{\cos x}}+\frac{\cos x}{\underset{(\cos x)}{\sin x}}=\frac{\sin^2 x}{\sin x\cdot\cos x}+\frac{\cos^2 x}{\sin x\cdot\cos x}=\frac{\overbrace{\sin^2 x+\cos^2 x}^{1}}{\sin x\cdot\cos x}$$

$$=\frac{1}{\sin x\cdot\cos x}=\underbrace{\frac{1}{\sin x}}_{\operatorname{cosec} x}\cdot\underbrace{\frac{1}{\cos x}}_{\sec x}=\operatorname{cosec} x\cdot\sec x$$

EXAMPLE 24 Verify that $\dfrac{\tan x}{\sec x}=\sin x$.

Solution

$$\frac{\overbrace{\dfrac{\sin x}{\cos x}}^{\tan x}}{\underbrace{\dfrac{1}{\cos x}}_{\sec x}}=\frac{\sin x}{\cancel{\cos x}}\cdot\frac{\cos x}{1}=\sin x$$

EXAMPLE 25 Verify that

$$(1-\sin x)(1+\sin x)=\frac{1}{\sec^2 x}.$$

Solution

$$(1-\sin x)(1+\sin x)=1^2-\sin^2 x=1-\sin^2 x=\cos^2 x=\frac{1}{\dfrac{1}{\cos^2 x}}=\frac{1}{\left(\dfrac{1}{\cos x}\right)^2}$$

$$=\frac{1}{\sec^2 x}\qquad\left(a=\frac{1}{\dfrac{1}{a}},\ a\neq 0\right)$$

EXAMPLE 26 Simplify

$(1 - \sin 2\ 18°) \times \dfrac{\tan 35°}{\sin^2 72°} \cdot \dfrac{\sec 12°}{\cot 55°} \cdot \sin 78°.$

Solution

$$\underbrace{(1-\sin^2 18°)}_{\cos^2 18°} \cdot \frac{\tan 35°}{\underbrace{\sin^2 72°}_{\cos^2 18°}} \cdot \frac{\overbrace{\sec 12°}^{\frac{1}{\cos 12°}}}{\underbrace{\cot 55°}_{\tan 35°}} \cdot \underbrace{\sin 78°}_{\cos 12°} = \cancel{\cos^2 18°} \cdot \frac{\cancel{\tan 35°}}{\cancel{\cos^2 18°}} \cdot \frac{1}{\cancel{\cos 12°}} \cdot \frac{1}{\cancel{\tan 35°}} \cdot \cancel{\cos 12°}$$

$$= 1$$

E. FINDING A TRIGONOMETRIC RATIO FROM A GIVEN RATIO

Sometimes we are given one trigonometric ratio and we need to find another trigonometric ratio in the same triangle. Look at the steps we can use for problems like this.

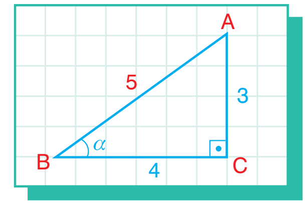
Property

1. Draw a right triangle and assign the angle in question to any one of the acute angles.
2. Use the given trigonometric ratio to write the lengths of the sides of the triangle.
3. Use the Pythagorean Theorem to find the length of the missing side.
4. Write the desired ratio by using the side lengths of the triangle.

EXAMPLE 27 Find $\sin\alpha$ given $\tan\alpha = \dfrac{3}{4}$.

Solution Follow the steps.

1. Draw the triangle opposite. Let us say assign m∠B = a.



2. $\tan\alpha = \frac{3}{4}, \frac{|AC|}{|BC|} = \frac{3}{4}$

   So $|AC| = 3$ and $|BC| = 4$.

3. $AB|^2 = |AC|^2 + |BC|^2$

   $|AB|^2 = 3^2 + 4^2$

   $|AB|^2 = 9 + 16$

   $|AB|^2 = 25$. So $|AB| = 5$.

4. $\sin\alpha = \frac{|AC|}{|AB|} = \frac{3}{5}$

EXAMPLE 28 Find tan x + cot x if $\cos x = \frac{5}{13}$.

Solution

$$|AC|^2 + |BC|^2 = |AB|^2$$
$$|AC|^2 + 5^2 = 13^2$$
$$|AC|^2 = 169 - 25$$
$$|AC|^2 = 144$$
$$|AC| = 12$$

So $\tan x = \frac{12}{5}$ and $\cot x = \frac{5}{12}$.

Therefore, $\tan x + \cot x = \underset{(12)}{\frac{12}{5}} + \underset{(5)}{\frac{5}{12}} = \frac{144+25}{5 \cdot 12} = \frac{169}{60}$.

EXAMPLE 29 $\dfrac{4\sin x + 3\cos x}{2\sin x - \cos x} = 5$ is given.

Find the ratios.

a. tan x

b. cot x

c. sin x

d. cos x

Solution First let us simplify the equation.

$$\frac{4\sin x + 3\cos x}{2\sin x - \cos x} = \frac{5}{1}$$

$$1 \times (4 \sin x + 3 \cos x) = 5 \times (2 \sin x - \cos x)$$

$$4 \sin x + 3 \cos x = 10 \sin x - 5\cos x$$

$$3 \cos x + 5 \cos x = 10 \sin x - 4\sin x$$

$$8 \cos x = 6 \sin x$$

a.

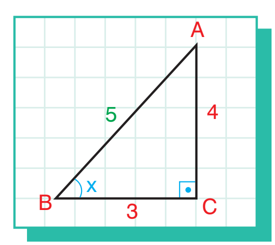
$$\frac{8\cos x}{\cos x} = \frac{6\sin x}{\cos x} \;\Rightarrow\; 8 = 6 \cdot \frac{\sin x}{\cos x} \;\Rightarrow\; \frac{8}{6} = \frac{\sin x}{\cos x}$$

So $\tan x = \dfrac{8}{6} = \dfrac{4}{3}$.

b.

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

c. Let us draw the triangle and find $|AB|$.



$|AB|^2 = 3^2 + 4^2$

$|AB|^2 = 25$

$|AB| = 5$

So $\sin x = \dfrac{4}{5}$.

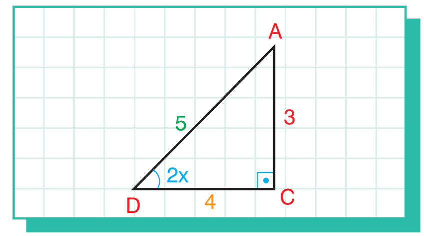
d.

$$\cos x = \frac{3}{5}$$

EXAMPLE 30 Find sin x if $\sin 2x = \dfrac{3}{5}$

Solution The question gives us a ratio for a right triangle with an angle 2x. We need to make a right triangle with an angle x. Look at the first figure.

Let us apply the Pythagorean Theorem to DADC:



$$|AD|^2 = |AC|^2 + |DC|^2$$

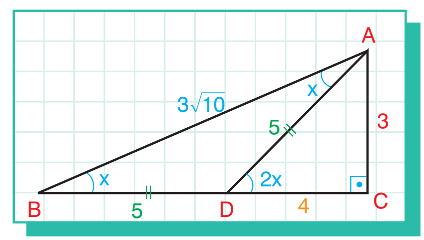
$$5^2 = 3^2 + |DC|$$

$$|DC| = 4.$$

Now let us draw [BD] which is congruent to [AD] as shown in the second figure (points B, D, and C are collinear).

If we draw [AB], then $\angle ABD = m\angle BAD = x$.

Apply the Pythagorean Theorem again to ΔABC:



$$|AB|^2 = |BC|^2 + |AC|^2$$

$$|AB|^2 = 9^2 + 3^2$$

$$|AB|^2 = 90$$

$$|AB| = \sqrt{90}$$

$$|AB| = 3\sqrt{10}.$$

$$\text{So } \sin x = \frac{\cancel{3}}{\cancel{3}\sqrt{10}} = \underset{(\sqrt{10})}{\frac{1}{\sqrt{10}}} = \frac{\sqrt{10}}{10}.$$

F. RATIOS IN A 30°-60°-90° TRIANGLE

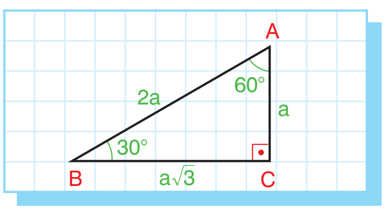
Look at the lengths of the sides of the triangle on the left.

We can write, $\sin 30° = \cos 60° = \dfrac{\cancel{a}}{2\cancel{a}} = \dfrac{1}{2}$

$$\sin 60° = \cos 30° = \frac{\cancel{a}\sqrt{3}}{2\cancel{a}} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \cot 60° = \frac{\cancel{a}}{\cancel{a}\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

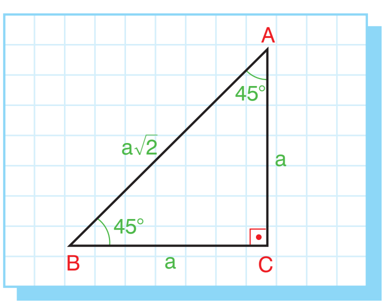
$$\cot 30° = \tan 60° = \frac{\cancel{a}\sqrt{3}}{\cancel{a}} = \sqrt{3}.$$



Objectives

After studying this section you will be able to give the trigonometric ratios of some common angles, and use them to solve problems.

G. RATIOS IN A 45°-45°-90° TRIANGLE



Similarly, by using the triangle on the left we can write,

$$\sin 45° = \cos 45° = \frac{\cancel{a}}{\cancel{a}\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \cot 45° = \frac{\cancel{a}}{\cancel{a}} = 1.$$

This gives us the values of the trigonometric ratios of some common angles.

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
cot	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$

EXAMPLE 31 Evaluate $\dfrac{4\cdot\sin 30°\cdot\tan 60°}{\tan 30°\cdot\cos 45°}$.

Solution Let us use the values from the table.

$$\frac{4\cdot\overbrace{\sin 30°}^{1/2}\cdot\overbrace{\tan 60°}^{\sqrt{3}}}{\underbrace{\tan 30°}_{\frac{\sqrt{3}}{3}}\cdot\underbrace{\cos 45°}_{\frac{\sqrt{2}}{2}}}=\frac{4}{\cancel{2}}\cdot\cancel{\sqrt{3}}\cdot\frac{3}{\cancel{\sqrt{3}}}\cdot\frac{\cancel{2}}{\sqrt{2}}=\frac{12}{\underset{(\sqrt{2})}{\sqrt{2}}}=\frac{12\sqrt{2}}{2}=6\sqrt{2}$$

EXAMPLE 32 Simplify $\dfrac{4+2\cdot\sin 30°}{\cot 30°}$.

Solution Let us use the values from the table.

$$\frac{4+2\cdot\overbrace{\sin 30°}^{1/2}}{\underbrace{\cot 30°}_{\sqrt{3}}}=\frac{4+\cancel{2}\cdot\frac{1}{\cancel{2}}}{\sqrt{3}}=\frac{5}{\underset{(\sqrt{3})}{\sqrt{3}}}=\frac{5\sqrt{3}}{3}$$

EXAMPLE 33 Simplify

$$3\cdot\frac{\sin 52°\cdot\tan 43°}{\cot 47°\cdot\cos 38°}-4\cdot\sin 60°\cdot\cos 60°.$$

Solution $$\frac{3\cdot\sin 52°\cdot\tan 43°}{\underbrace{\cot 47°}_{\tan 43°}\cdot\underbrace{\cos 38°}_{\sin 52°}}-4\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}=3-\sqrt{3}$$

EXAMPLE 34 Evaluate

$\tan 1°\times\tan 2°\times...\times\tan 88°\times\tan 89°$.

Solution Remember that $\tan a \times \cot a = 1$.

By using complementary angles we have

$\tan 89° = \cot 1°$, $\tan 88° = \cot 2°$, ... , $\tan 46° = \cot 44°$.

So we have

$\tan 1° \times \tan 2° \times ... \times \tan 88° \times \tan 89$

$= \tan 1° \times \tan 2° \times ... \times \tan 44° \times \tan 45° \times \cot 44° \times ... \times \cot 2° \times \cot 1°$

$= \underbrace{(\tan 1° \times \cot 1°)}_{1} \times \underbrace{(\tan 2° \times \cot 3°)}_{1} \times ... \times \underbrace{(\tan 44° \times \cot 44°)}_{1} \times \tan 45°$

$= \tan 45°$

$= 1.$

Activity

1. Simplify the ratios.

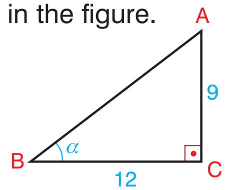
   a. $\dfrac{\tan 30°\cdot\cot 60°}{\sin 30°}$ b. $\dfrac{3+2\cdot\sin 30°}{1-\sqrt{3}\cdot\tan 60°}$

2. Evaluate $\dfrac{\sin 30°\cdot\sin 60°+\cos 30°\cdot\cos 60°}{\tan 30°\cdot\tan 60°+\tan 45°}$.

3. Evaluate $\cot 1° \times \cot 2° \times \cot 3° \times ... \times \cot 88° \times \cot 89°$.

EXERCISES 2.1

1. Write the ratios for the triangle in the figure.



a. $\sin\alpha$ **b.** $\cos\alpha$

c. $\tan\alpha$ **d.** $\cot\alpha$

e. $\sec\alpha$ **f.** $\operatorname{cosec}\alpha$

g. $\sin(90° - \alpha)$ **h.** $\cot(90° - \alpha)$

2. Simplify the ratios.

a. $\sin^2 x \cdot \cot^2 x$

b. $\cot^2 x \cdot \sec^2 x \cdot \sin x$

c. $(\sin x + \cos x \cdot \cot x) \cdot \tan x$

d. $\cot x \cdot (\tan x + \cot x)$

e. $\dfrac{1+\tan^2 x}{\tan^2 x}$

f. $\dfrac{1-\operatorname{cosec}^2 x}{1-\sin^2 x}$

g. $\dfrac{1}{\sin x \cdot \cos x} - \cot x$

3. Find the value of x in each equation.

a. $\tan x = \cot 73°$

b. $\sin 2x = \cos 66°$

c. $\cos(x - 10°) = \sin 70°$

d. $\cot(5x + 5°) = \tan 15°$

4. Verify each equation.

a. $\sin\alpha \cdot (\operatorname{cosec}\alpha - \sin\alpha) = \cos^2\alpha$

b. $\sin\alpha \cdot \tan\alpha + \cos\alpha = \sec\alpha$

c. $(\sin\alpha - \cos\alpha)^2 + (\sin\alpha + \cos\alpha)^2 = 2$

d. $\dfrac{1+\cot^2 x}{1+\tan^2 x} = \cot^2 x$

e. $\dfrac{\sin x}{\operatorname{cosec} x} + \dfrac{\cos x}{\sec x} = 1$

5. Simplify $\dfrac{\tan 27° \cdot (\sin^2 13° + \cos^2 13°)}{(\sec^2 5° - \tan^2 5°) \cdot \cot 63°}$.

6. Simplify $\dfrac{\sin 5° \cdot \sin 10° \cdot \sin 15° \cdot \sin 20°}{\cos 70° \cdot \cos 75° \cdot \cos 80° \cdot \cos 85°}$.

7. Simplify $\cot 5° \cdot \cot 10° \cdot \cot 15° \cdot \ldots \cdot \cot 85°$.

8. Simplify $\dfrac{5 - 5\cos^2 x}{\tan^2 x \cdot \cos^2 x}$.

9. Find the values.

a. $\sin 45° \cdot \cos 30° \cdot \tan 60° \cdot \cot 45°$

b. $\tan 60° \cdot \cot 60° + \sin^2 60° + \cos^2 60°$

c. $\sec 30° + \cot 45° + \cos 30°$

10. Find $\sin\theta$, $\cos\theta$, $\tan\theta$ if $\cot\theta = \dfrac{24}{7}$.

11. Find $\dfrac{\sec\alpha + \cos\alpha}{\tan\alpha + \operatorname{cosec}\alpha}$ given $\sin\alpha = \dfrac{4}{5}$.

12. Find $\cot x$ if $\tan 2x = 2$.

13. Find the ratios using a trigonometric table.

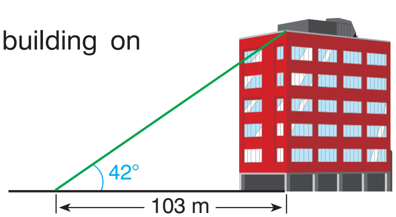
a. $\cos 17°$ **b.** $\tan 46°$ **c.** $\sin 78°$

14. Use a table of trigonometric ratios to find the approximate measure of $\angle A$.

a. $\sin A = 0.743$ **b.** $\cot A = 1.304$

c. $\cos A = 0.346$ **d.** $\tan A = 2.426$

15. How tall is the building on right?



16. A plane makes an angle of depression of 33° with a runway. Its altitude is 5200 m. Find the horizontal distance from the plane to the runway.

1. Distance Between Two Points

Let us use x_0, x_1, x_2, ... and y_0, y_1, y_2, ... to denote the abscissas and the ordinates of respective points in the coordinate plane.

Theorem

distance between two points

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\boxed{AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

Proof In the figure, ΔABC is a right triangle.

$AC = x_2 - x_1$

$BC = y_2 - y_1$

By the Pythagorean theorem,

$$AB^2 = AC^2 + BC^2$$

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

and so $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ or

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

EXAMPLE 35 Find the distance between $A(3, 0)$ and $B(-2, -3)$.

Solution $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2-3)^2 + (-3-0)^2} = \sqrt{(-5)^2 + (-3)^2} = \sqrt{34}$ units.

EXAMPLE 36 Show that ΔABC with the vertices $A(-2, 2)$, $B(1, 5)$, and $C(4, -1)$ is an isosceles triangle.

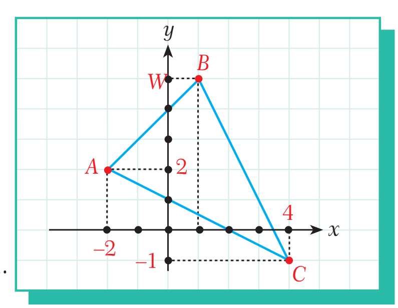
Solution Let us find the length of the sides of ΔABC.

$$AB = \sqrt{(1+2)^2 + (5-2)^2} = \sqrt{9+9} = 3\sqrt{2}$$

$$AC = \sqrt{(4+2)^2 + (-1-2)^2} = \sqrt{36+9} = 3\sqrt{5}$$

$$BC = \sqrt{(4-1)^2 + (-1-5)^2} = \sqrt{9+36} = 3\sqrt{5}$$

$AC = BC$, so two sides of the triangle have the same length. Therefore, ΔABC is isosceles.



EXAMPLE 37 $A(a, 2)$, $B(3, 4)$, and $C(-2, 1)$ are given. If A is at the same distance from the points B and C, find a.

Solution We are given $AB = AC$. By the theorem for the distance between two points,

$$\sqrt{(3-a)^2+2^2} = \sqrt{(a+2)^2+1^2}$$

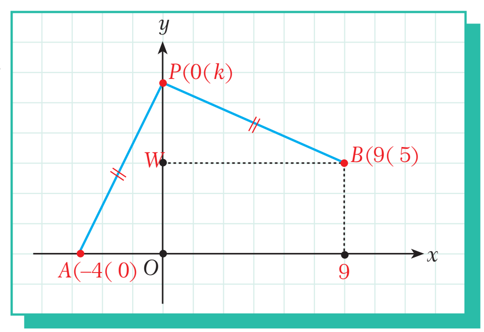
$$9-6a+\cancel{a^2}+4 = \cancel{a^2}+4a+4+1$$

$$10a = 8$$

$$a = \frac{4}{5}.$$

EXAMPLE 38 Find the ordinate of the point on the y-axis which is equidistant to the points $A(-4, 0)$ and $B(9, 5)$.

Solution The point is on the y-axis, so its x-coordinate is 0. Let us call the point $P(0, k)$. Now, from the diagram,



$$PA = PB$$

$$\sqrt{(-4)^2+k^2} = \sqrt{9^2+(k-5)^2}$$

$$(-4)^2 + k^2 = 9^2 + (k-5)^2$$

$$16 + k^2 = 81 + k^2 - 10k + 25$$

$$10k = 90$$

$k = 9$. Therefore, the point is $P(0, 9)$.

Check Yourself

1. Find the distance between the points $A(2, -1)$ and $B(-2, 2)$.
2. Find the lengths of the sides of the triangle MNP with vertices at the points $M(-1, 3)$, $N(-2, -3)$, and $P(5, 1)$.
3. The points $K(2, 1)$ and $L(-6, a)$ are given. If $KL = 10$ cm, find the possible values of a.
4. A is a point on the y-axis with ordinate 5 and B is the point $(-3, 1)$. Calculate AB.
5. Find the point on the y-axis which is equidistant to the points $A(-3, 0)$ and $B(4, -1)$.

Answers

1. 5 2. $\sqrt{37}$, $2\sqrt{10}$, $\sqrt{65}$ 3. $a \in \{-5, 7\}$ 4. 5 5. $(0, -4)$

2. Midpoint of a Line Segment

Theorem

midpoint of a line segment

Let the points $A(x_1, y_1)$ and $B(x_2, y_2)$ be the endpoints of a line segment AB, and let $C(x_0, y_0)$ be the midpoint of AB. Then,

$$\boxed{x_0 = \frac{x_1 + x_2}{2} \quad \text{and} \quad y_0 = \frac{y_1 + y_2}{2}}.$$

Proof Let us take point C on AB such that $AC = CB$.

From the figure, $\Delta CAK \cong \Delta BCD$.

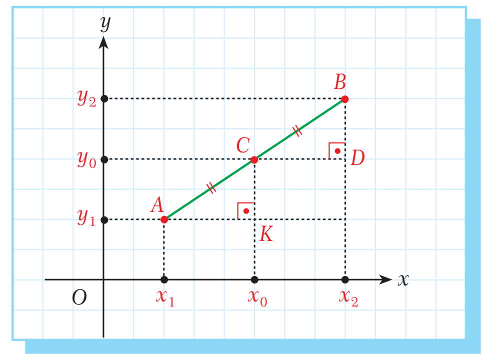
So $AK = CD$ and $CK = BD$.

Now, $x_0 - x_1 = x_2 - x_0$ and $y_0 - y_1 = y_2 - y_0$

$2x_0 = x_1 + x_2$ and $2y_0 = y_1 + y_2$

$x_0 = \dfrac{x_1 + x_2}{2}$ and $y_0 = \dfrac{y_1 + y_2}{2}$.

So $C(x_0, y_0) = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.



EXAMPLE 39 $A(-1, -2)$ and $B(-5, 4)$ are given. Find the coordinates of the midpoint of AB.

Solution $x_0 = \dfrac{x_1 + x_2}{2} = \dfrac{-1-5}{2} = -3$, $y_0 = \dfrac{y_1 + y_2}{2} = \dfrac{-2+4}{2} = 1$

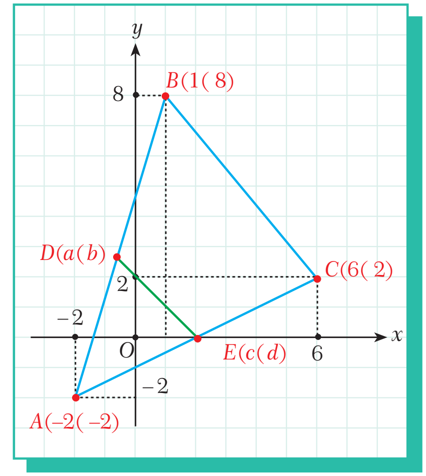
So $C(-3, 1)$ is the midpoint of AB.

EXAMPLE 40 A triangle ABC with vertices $A(-2, -2)$, $B(1, 8)$, and $C(6, 2)$ is given. If the points D and E are midpoints of AB and AC respectively, show that $ED = \dfrac{BC}{2}$.

Solution First, let us find the coordinates of $D(a, b)$ and $E(c, d)$. Points $D(a, b)$ and $E(c, d)$ are the midpoints of AB and AC, so their coordinates are

$$\left.\begin{aligned} a &= \frac{x_1 + x_2}{2} = \frac{-2+1}{2} = -\frac{1}{2} \\ b &= \frac{y_1 + y_2}{2} = \frac{-2+8}{2} = 3 \end{aligned}\right\} \Rightarrow D(-\frac{1}{2}, 3),$$

$$\left.\begin{aligned} c &= \frac{x_1+x_2}{2} = \frac{-2+6}{2} = 2 \\ d &= \frac{y_1+y_2}{2} = \frac{-2+2}{2} = 0 \end{aligned}\right\} \Rightarrow \quad E(2, 0).$$



Now, let us find the length of ED and BC by using the distance formula, and then compare their lengths:

$$ED = \sqrt{(2+\frac{1}{2})^2 + (0-3)^2}$$

$$= \sqrt{\frac{25}{4}+9} = \sqrt{\frac{61}{4}} = \frac{\sqrt{61}}{2}$$

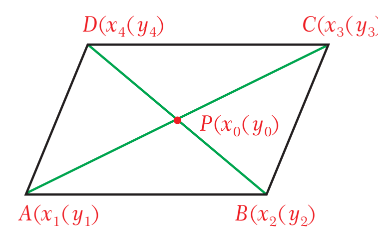
$$BC = \sqrt{(6-1)^2 + (2-8)^2}$$

$$= \sqrt{25+36} = \sqrt{61}.$$

Hence, $ED = \dfrac{BC}{2}$.

Rule

Let the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, and $D(x_4, y_4)$ be vertices of a parallelogram $ABCD$, and let $P(x_0, y_0)$ be the intersection point of the diagonals.
Since $P(x_0, y_0)$ is the midpoint of the diagonals,



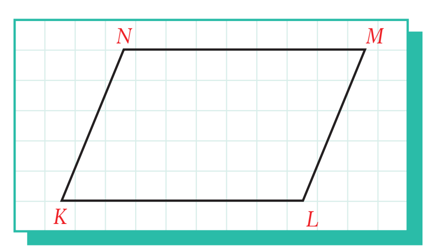
$$x_0 = \frac{x_1+x_3}{2} \text{ and } x_0 = \frac{x_2+x_4}{2}, \text{ so } x_1+x_3 = x_2+x_4.$$

$$y_0 = \frac{y_1+y_3}{2} \text{ and } y_0 = \frac{y_2+y_4}{2}, \text{ so } y_1+y_3 = y_2+y_4.$$

As a result, for any parallelogram $ABCD$ with given vertices the following rules are valid:

$$\boxed{x_1 + x_3 = x_2 + x_4 \text{ and } y_1 + y_3 = y_2 + y_4}.$$

EXAMPLE 41 $KLMN$ is a parallelogram with vertices $K(2, a)$, $L(1, 4)$, $M(b, 3)$, and $N(3, 2)$. Find $a - b$.



Solution The midpoint of KM is also the midpoint of NL, so

$$2 + b = 1 + 3 \quad \text{and} \quad a + 3 = 4 + 2$$
$$b = 2 \qquad\qquad a = 3.$$

Therefore, $a - b = 3 - 2 = 1$.

Check Yourself

1. $A(a + 1, 4 - 2b)$ and $B(3 - a, 2b - 3)$ are given. Find the coordinates of the midpoint of AB.
2. A triangle ΔABC with vertices $A(2, 5)$, $B(-2, 3)$, and $C(4, -1)$ is given. Find the length of the median passing through A.
3. The points $A(-2, -3)$, $B(3, -2)$, $C(x, y)$, and $D(-1, 3)$ are the vertices of a parallelogram $ABCD$. Find the coordinates of C.

Answers

1. $(2, \frac{1}{2})$ 2. $\sqrt{17}$ 3. $(4, 4)$

Properties 4

Triangle Inequality Theorem

In any triangle ABC with sides a, b and c, the following inequalities are true:

$$|b - c| < a < (b + c),$$
$$|a - c| < b < (a + c),$$
$$|a - b| < c < (a + b).$$

The converse is also true. This property is also called the **Triangle Inequality Theorem**.

EXAMPLE 42 Is it possible for a triangle to have sides with the lengths indicated?

a. 7, 8, 9 b. 0.8, 0.3, 1 c. $\frac{1}{2}, \frac{1}{3}, 1$

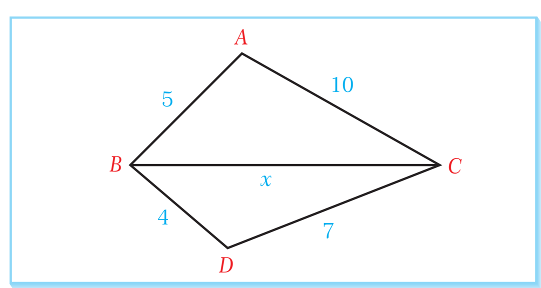
Solution We can check each case by using the Triangle Inequality Theorem.

a. $|9 - 8| < 7 < (8 + 9)$
$|8 - 9| < 8 < (7 + 9)$
$|7 - 8| < 9 < (7 + 8)$.
This is true, so by the Triangle Inequality Theorem this is a possible triangle.

b. $|0.8 - 0.3| < 1 < (0.8 + 0.3)$
$|1 - 0.3| < 0.8 < (1 + 0.3)$
$|1 - 0.8| < 0.3 < (1 + 0.8)$.
This is true, so by the Triangle Inequality Theorem this is a possible triangle.

c. This is impossible, since
$1 \not< \frac{1}{2} + \frac{1}{3}$.

EXAMPLE 43 Find all the possible integer values of x in the figure.



Solution In ΔABC, $|10 - 5| < x < (10 + 5)$ (Triangle Inequality Theorem)

$5 < x < 15.$ (1)

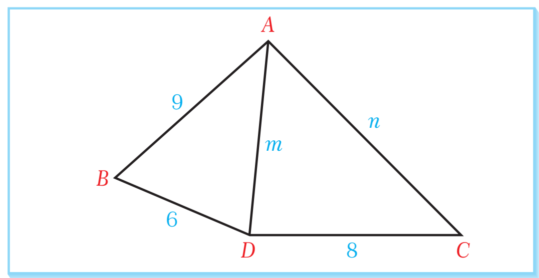
In ΔDBC, $|7 - 4| < x < (7 + 4)$ (Triangle Inequality Theorem)

$3 < x < 11.$ (2)

The possible values of x are the elements of the common solution of inequalities (1) and (2), i.e. $5 < x < 11$.

So $x \in \{6, 7, 8, 9, 10\}$.

EXAMPLE 44 Find the greatest possible integer value of m in the figure, then find the smallest possible integer value of n for this case.



Solution In ΔABD, $|9 - 6| < m < (9 + 6)$ (Triangle Inequality Theorem)

$3 < m < 15.$

So the greatest possible integer value of m is 14.

In ΔADC, $|8 - m| < n < (m + 8)$ (Triangle Inequality Theorem)

$|8 - 14| < n < (14 + 8)$ $(m = 14)$

$6 < n < 22.$

So when $m = 14$, the smallest possible integer value of n is 7.

EXAMPLE 45 In a triangle ABC, $m(\angle A) > 90°$, $c = 6$ and $b = 8$. Find all the possible integer lengths of a.

Solution Since $m(\angle A) > 90°$, $\sqrt{b^2 + c^2} < a < (b + c)$ by Property 5.1.

Substituting the values in the question gives $\sqrt{8^2 + 6^2} < a < (8 + 6)$, i.e.

$10 < a < 14$. So $a \in \{11, 12, 13\}$.

Check Yourself

1. Two sides of a triangle measure 24 cm and 11 cm respectively. Find the perimeter of the triangle if its third side is equal to one of other two sides.
2. Determine whether each ratio could be the ratio of the lengths of the sides of a triangle.
   a. 3 : 4 : 5   b. 4 : 3 : 1   c. 10 : 11 : 15   d. 0.2 : 0.3 : 0.6
3. The lengths of the sides DE and EF of a triangle DEF are 4.5 and 7.8. What is the greatest possible integer length of DF?
4. The base of an isosceles triangle measures 10 cm and the perimeter of the triangle is an integer length. What is the smallest possible length of the leg of this triangle?
5. In an isosceles triangle KLM, $KL = LM = 7$ and $m(\angle K) < 60°$. If the perimeter of the triangle is an integer, how many possible triangle(s) KLM exist?
6. In a triangle ABC, $AB = 9$ and $BC = 12$. If $m(\angle B) < 90°$, find all the possible integer lengths of AC.

Answers

1. 59 cm   2. a. yes b. no c. yes d. no   3. 12   4. 5.5 cm   5. six triangles
6. $AC \in \{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$

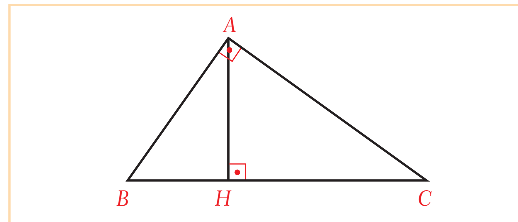
H. EUCLIDEAN RELATIONS

Theorem

The altitude to the hypotenuse of a right triangle divides the right triangle into two smaller right triangles which are similar to the original triangle, and therefore also similar to each other.

Proof

Look at the first figure.



Given: ΔABC is a right triangle and AH is the altitude to the hypotenuse.

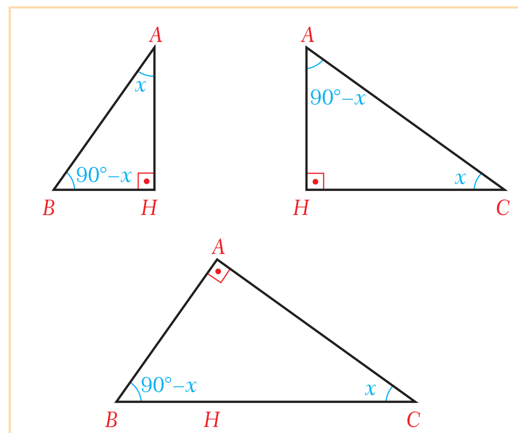
Prove: $\Delta ABC \sim \Delta HBA \sim \Delta HAC$

We will give the proof in paragraph form.

Let $m(\angle BCA) = x$.

Then, $m(\angle ABC) = 90° - x$, $m(\angle HAB) = x$ and $m(\angle HAC) = 90° - x$.

So each smaller triangle is similar to the larger triangle by the AA Similarity Theorem, and therefore the two smaller triangles are also similar to each other.



Remember!

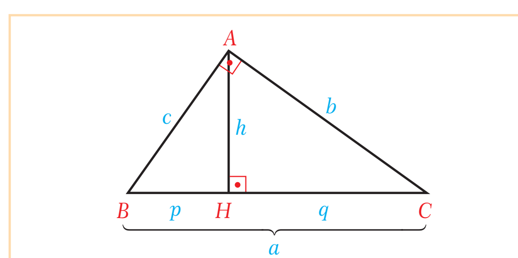
The geometric mean of two numbers a and b is a positive number x such that $\frac{a}{x} = \frac{x}{b}$, i.e. $x = \sqrt{a \cdot b}$.

This theorem leads us to two more useful theorems.

Theorem

Euclidean theorems

In any right triangle, when the altitude to the hypotenuse is drawn, the following two statements are true:



1. The length of the altitude is the geometric mean of the lengths of the two segments of the hypotenuse formed by the altitude ($AH^2 = BH \cdot CH$ in the figure).
2. The length of each leg is the geometric mean of the length of its adjacent hypotenuse segment and the length of the hypotenuse. ($CA^2 = CH \cdot CB$ in the figure).

Proof Let us draw an appropriate figure (shown at the right).

Given: ΔABC is a right triangle and AH is the altitude to the hypotenuse.

Prove: $AH^2 = BH \cdot CH$ (1) and

$BA^2 = BH \cdot BC$ and $CA^2 = CH \cdot CB$ (2)

We will write the proof of (1) in paragraph form.

By the theorem at the beginning of this section, $\Delta AHB \sim \Delta CHA$.

By the definition of similarity, corresponding sides are proportional:

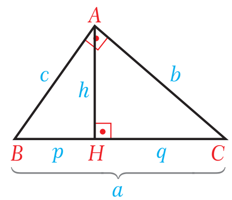
$$\frac{BH}{AH} = \frac{AH}{CH}, \text{ i.e. } AH^2 = BH \cdot CH, \text{ as required.}$$

Now let us prove (2). By the same theorem, $\Delta HBA \sim \Delta ABC$. So by the definiton of similarity, corresponding sides are proportional:

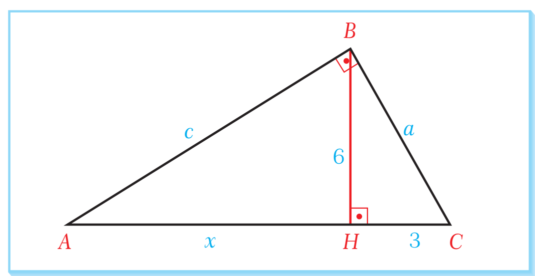
$$\frac{BH}{BA} = \frac{BA}{BC}, \text{ i.e. } BA^2 = BH \cdot BC.$$

By a similar argument, $\Delta HAC \sim \Delta ABC$. So $\dfrac{CH}{CA} = \dfrac{CA}{CB}$, i.e. $CA^2 = CH \cdot CB$.

For any right triangle ABC, the relations $h^2 = p \cdot q$, $c^2 = p \cdot a$ and $b^2 = q \cdot a$ are also called **Euclidean relations.**



EXAMPLE 46 Find the lengths a, c and x in the figure.



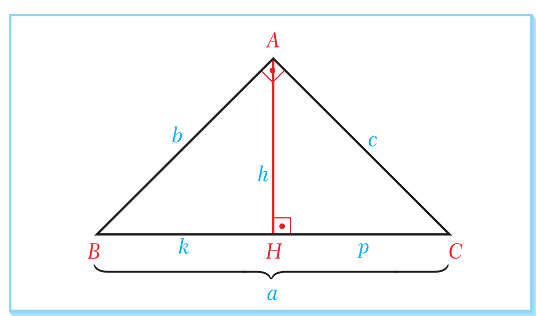
Solution Since ΔABC is a right triangle and BH is an altitude, we can use the Euclidean relations:

$h^2 = p \cdot q$; $6^2 = x \cdot 3$; $x = 12$,

$a^2 = 3 \cdot (3 + 12) = 3 \cdot 36$; $a = 6\sqrt{3}$,

$c^2 = 12 \cdot (12 + 3) = 180$; $c = 6\sqrt{5}$.

EXAMPLE 47 Prove that $\dfrac{1}{h^2} = \dfrac{1}{b^2} + \dfrac{1}{c^2}$ in the figure.



AREAS OF QUADRILATERALS

A. THE CONCEPT OF AREA

1. Basic Definitions

Definition

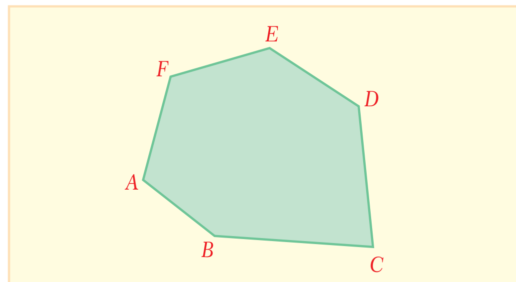
polygonal region

The union of a polygon and its interior region is called a **polygonal region**.

sides	name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon
⋮	⋮

We name polygons by their vertices. For example, ΔABC is the name of a triangle with vertices at points A, B and C, and $ABCD$ is the name of a quadrilateral with vertices at points A, B, C and D. We use extra notation to refer to a polygonal region: (ΔABC) is a triangular region, and $(ABCDE)$ is a pentagonal region.

In the figure, $(ABCDEF)$ is the union of the hexagon and its interior region. Since $ABCDEF$ is a hexagon, we can say that $(ABCDEF)$ is a hexagonal region.

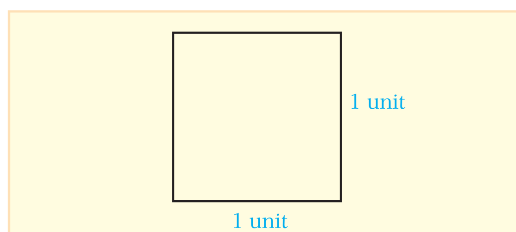


Definition

square unit

The interior region of a square with side length one unit is called a **square unit**. We write **unit**^{2} to mean a square unit.

In the figure opposite, each side of the square measures 1 unit and so its area is 1 square unit, or 1 unit^2. We can also use metric units for lengths and areas: a square with side 1 cm has area 1 cm^2, and a square with side 1 m has area 1 m^2, etc.



Definition

area

The **area** of a closed plane figure is the total number of non-overlapping square units and part units that cover the surface of the polygonal region. The area of a figure is always a positive real number.

We use the letter ***A*** to mean the area of a polygon: the area of ΔABC is $A(\Delta ABC)$, and the area of the pentagon $ABCDE$ is $A(ABCDE)$.

If the sides of a figure are not natural numbers or if the polygon is very big, it is difficult to find its area by counting the individual unit squares. In this book we will learn a set of formulas and methods to find the area of any geometric figure by calculation.

Definition

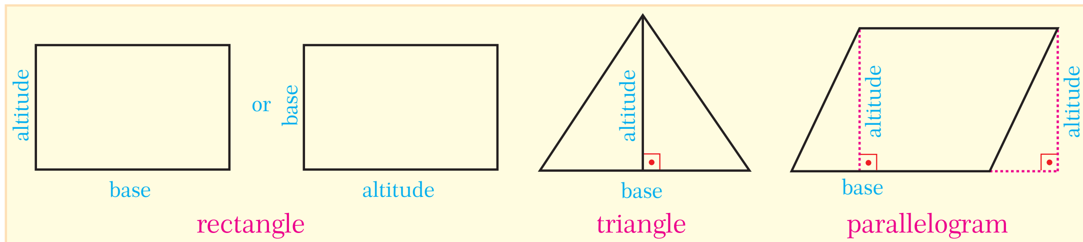
altitude, height

An **altitude** is a line segment between a vertex of a polygon and a line containing a side of the polygon, which is perpendicular to this line. The length of an altitude is called a **height** of the polygon. We write h_a, h_b, etc. to mean the altitudes to sides a, b, etc. of a polygon.

Definition

base

The side of a polygon from which we draw an altitude is called the **base**. We can use any side of a polygon as a base.



Note

In an isosceles triangle, the congruent sides are called the legs of the triangle and the third side is the base.

Postulate

area congruence postulate

If two figures are congruent then their areas are the same.

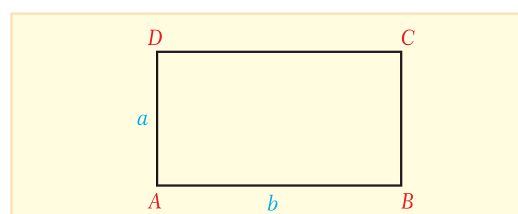
Two polygons are congruent if their corresponding sides and angles are the same.

2. Area of a Rectangle

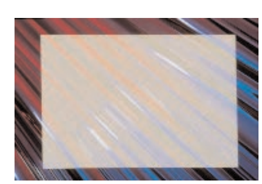
Postulate

The area of a rectangle is the product of the lengths of two consecutive sides:

$$\boxed{A(ABCD) = a \cdot b}\ .$$



EXAMPLE 1 Two sides of a rectangle measure 14 cm and 20 cm. What is the area of this rectangle?



Solution Let the sides of the rectangle be $a = 14$ cm and $b = 20$ cm. Then

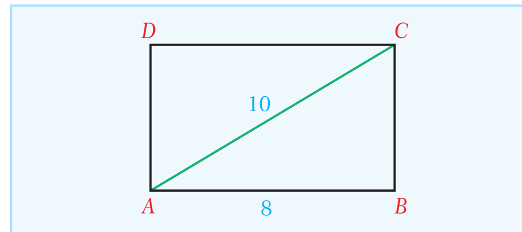
$$A = a \cdot b = 14 \cdot 20 = 280 \text{ cm}^2.$$

EXAMPLE 2 A rectangle has area 84 cm^2 and one of its sides measures 7 cm. What is this perimeter of this rectangle?

Solution Let us write $a = 7$ cm and $A = 84$ cm^2. So $A = a \cdot b$ gives us $84 = 7 \cdot b$, i.e. $b = 12$ cm.

So the perimeter of the rectangle is $2(a + b) = 2(7 + 12) = 38$ cm.

EXAMPLE 3 In the figure, $ABCD$ is a rectangle with $AC = 10$ cm and $AB = 8$ cm. Find the area of this rectangle.



Solution We can use the Pythagorean Theorem or special right triangles to find the length of $BC = b$. By the Pythagorean Theorem,

$$AB^2 + BC^2 = AC^2$$
$$8^2 + b^2 = 10^2$$
$$b = 6 \text{ cm.}$$

So $A(ABCD) = a \cdot b = 6 \cdot 8 = 48$ cm^2.

EXAMPLE 4 A rectangle has area 108 cm^2 and one of its sides measures three times the other side. Find the perimeter of this rectangle.

Solution Let $a = x$ and $b = 3x$ be the side lengths since one side is three times as long as the other side.

Then $A = ab$ gives us $108 = x \cdot 3x = 3x^2$, so $x^2 = 36$ and $x = 6$.

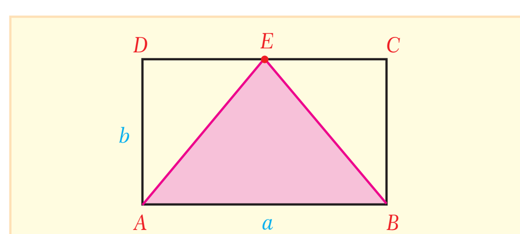
So $a = 6$ cm and $b = 18$ cm.

So the perimeter of $ABCD$ is $2 \cdot (a + b) = 2 \cdot (6 + 18) = 48$ cm.

Rule

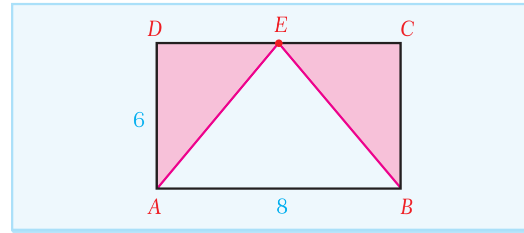
If we connect a point on one side of a rectangle to the endpoints of the opposite side then the area of the triangle obtained is half the area of the rectangle: in the figure,

$$\boxed{A(\Delta ABE) = \frac{A(ABCD)}{2} = \frac{a \cdot b}{2}}.$$



EXAMPLE 5 In the figure, $ABCD$ is a rectangle with sides $AB = 8$ cm and $AD = 6$ cm, and E is a point on side DC.

Find the combined area of the shaded regions.



Solution By the previous rule, $A(\Delta ABE) = \dfrac{A(ABCD)}{2}$.

So the sum of the shaded areas will also be $\dfrac{A(ABCD)}{2}$.

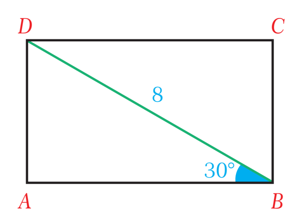
So the combined area is

$$\frac{A(ABCD)}{2} = \frac{6 \cdot 8}{2} = 24 \text{ cm}^2.$$

Check Yourself

1. A rectangle has perimeter 40 cm and one side is 4 cm longer than the other side. Find the area of this rectangle.

2. In the figure, $ABCD$ is a rectangle with $BD = 8$ cm and $m(\angle ABD) = 30°$. Find $A(ABCD)$.



3. A rectangle has area 48 cm^2 and perimeter 28 cm. Find the lengths of the sides of this rectangle.

4. One side of a rectangle is twice as long as another side. Given that the perimeter of this rectangle is 30 cm, find its area.

Answers

1. 96 cm^2 2. $16\sqrt{3}$ cm^2 3. 6 cm, 8 cm 4. 50 cm^2

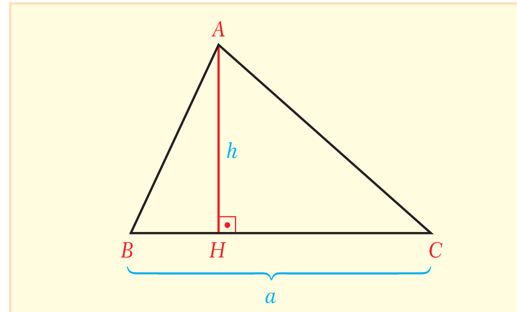
B. AREA OF A TRIANGLE

Theorem

base-height formula

The area of a triangle is half the product of the length of one base and the height of the triangle from that base: in the figure,

$$\boxed{A(\Delta ABC) = \frac{a \cdot h}{2}}.$$

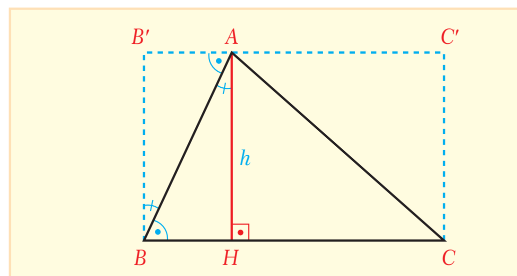


Proof

Look at the figure.

In ΔABC, $BC = a$.

We draw a line parallel to BC through A, and from B and C we draw perpendiculars BB' and CC' to the parallel line.



In a triangle, the side opposite vertex A is called a, the side opposite B is called b, and the side opposite C is called c.

We can say that $BCC'B'$, $BHAB'$ and $AHCC'$ are rectangles, and also $BB' = CC' = AH = h$.

Also, $\Delta ABH \cong \Delta BAB'$ since $\angle ABH \cong \angle BAB'$, $\angle BAH \cong \angle B'BA$ and AB is a common side.

By the Area Congruence Postulate we can write $A(\Delta ABH) = A(\Delta ABB') = X$.

Area Congruence Postulate: If two figures are congruent then their areas are the same.

By similar reasoning we have

$A(\Delta AHC) = A(\Delta ACC') = Y$.

So $A(BCC'B') = a \cdot h = 2X + 2Y$

$$= 2 \cdot (X + Y), \text{ i.e. } X + Y = \frac{a \cdot h}{2}.$$

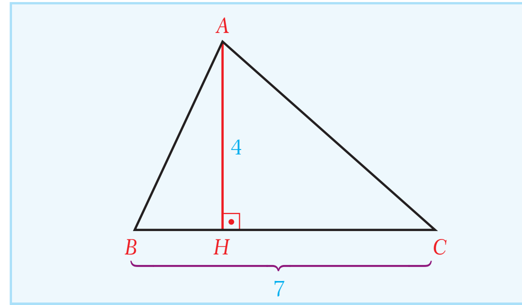
Finally, $A(\Delta ABC) = X + Y = \dfrac{a \cdot h}{2}$.

So $A(\Delta ABC) = \dfrac{a \cdot h}{2}$, as required.

Note

We can use any side of a triangle as a base, so $A(\Delta ABC) = \dfrac{a \cdot h_a}{2} = \dfrac{b \cdot h_b}{2} = \dfrac{c \cdot h_c}{2}$.

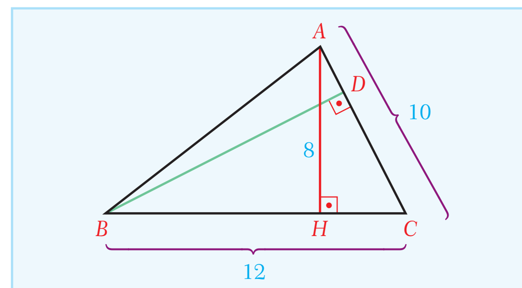
EXAMPLE 6 In the figure,
$BC = 7$ cm and
$AH = 4$ cm.
Find $A(\Delta ABC)$.



Solution In the figure, $BC = a = 7$ and
$AH = h = 4$.

So $A(\Delta ABC) = \dfrac{a \cdot h}{2} = \dfrac{7 \cdot 4}{2} = 14 \text{ cm}^2$.

EXAMPLE 7 In the triangle opposite,
$AH = 8$,
$BC = 12$ and
$AC = 10$.
Find the length of BD.



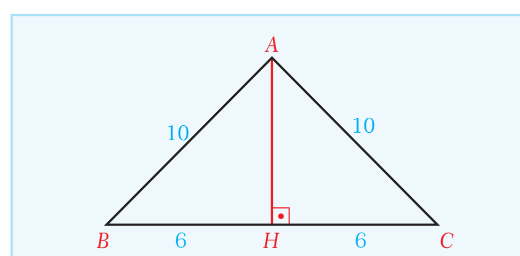
Solution $BC = a = 12$,
$AC = b = 10$,
$AH = h_a = 8$ and we need to find h_b.

We have $\dfrac{a \cdot h_a}{2} = \dfrac{b \cdot h_b}{2}$, so $\dfrac{12 \cdot 8}{2} = \dfrac{10 \cdot h_b}{2}$ and so $h_b = \dfrac{48}{5}$.

EXAMPLE 8 The base of an isosceles triangle measures 12 cm and the other sides are each 10 cm long. Find the area of this triangle.

Solution The figure shows the triangle with
$BC = a = 12$ cm.

We draw the altitude $AH = h_a$. Because the triangle is isosceles, H will be the midpoint of side BC.

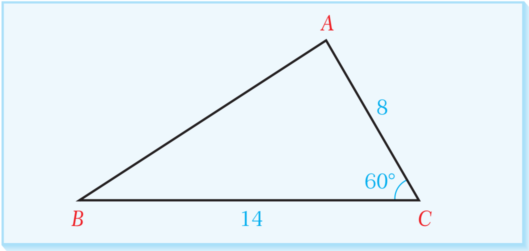


By the Pythagorean Theorem in ΔAHC,

$$h_a^2 + 6^2 = 10^2$$
$$h_a^2 = 100 - 36$$
$$h_a^2 = 64$$
$$h_a = 8 \text{ cm}.$$

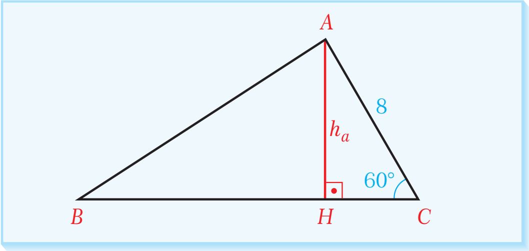
So $A(\Delta ABC) = \dfrac{a \cdot h_a}{2} = \dfrac{12 \cdot 8}{2} = 48 \text{cm}^2$.

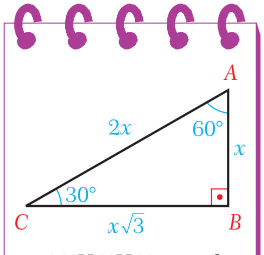
EXAMPLE 9 In the triangle ABC opposite, $BC = 14$ cm, $AC = 8$ cm and $m(\angle ACB) = 60°$. Find $A(\Delta ABC)$.



Solution We know $BC = a = 14$.

Let us draw $AH = h_a$.





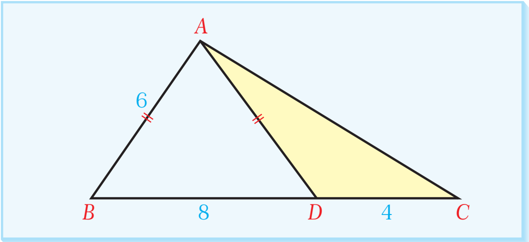
a 30°–60°–90° triangle

Then ΔAHC is a right triangle, and so we can use the properties of a 30°-60°-90° triangle: if the hypotenuse AC measures 8 cm then the side opposite the 60° angle measures

$$\frac{8\sqrt{3}}{2} = 4\sqrt{3} = h_a.$$

$$\text{So } A(\Delta ABC) = \frac{a \cdot h_a}{2} = \frac{14 \cdot 4\sqrt{3}}{2} = 28\sqrt{3}\ \text{cm}^2.$$

EXAMPLE 10 In the figure, $AB = AD = 6$ cm, $BD = 8$ cm and $DC = 4$ cm. Find $A(\Delta ADC)$.



Solution Let us draw the altitude AH to side BC and write $AH = h$.

ΔABD is isosceles, so $BH = HD = 4$ cm.

Now we can use the Pythagorean Theorem in ΔABH to find h:

$h^2 + 4^2 = 6^2$ and $h^2 = 20$, i.e. $h = 2\sqrt{5}$ cm.

$$\text{So } A(\Delta ADC) = \frac{AH \cdot DC}{2} = \frac{2\sqrt{5} \cdot 4}{2} = 4\sqrt{5}\ \text{cm}^2.$$

Check Yourself

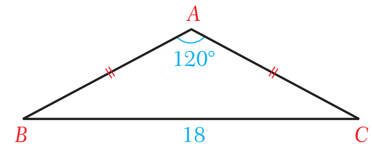
1. Find the area of the triangle with the given base and altitude.

   a. $a = 4$, $h_a = 7$ b. $b = 3$, $h_b = 8$

2. The sides of ΔABC are $a = 6$ cm, $b = 8$ cm and $c = 10$ cm, and $A(\Delta ABC) = 24$ cm^2. What are the three heights of this triangle?

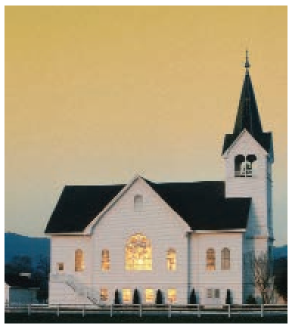
3. In a triangle, $a = 6$ cm and $c = 12$ cm. Find h_c if $h_a = 10$ cm.

4. In the isosceles triangle opposite, $AB = AC$, $m(\angle A) = 120°$ and the length of the base is $BC = 18$. Find $A(\Delta ABC)$.



Answers

1. a. 14 b. 12 2. $h_a = 8$ cm, $h_b = 6$ cm, $h_c = \frac{24}{5}$ cm 3. 5 cm 4. $27\sqrt{3}$

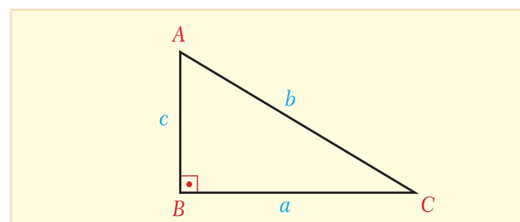


Theorem

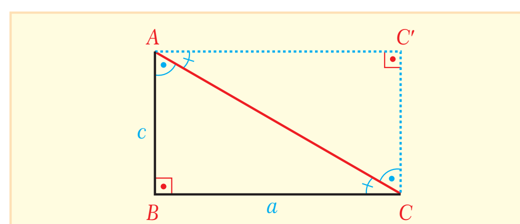
area of a right triangle

The area of a right triangle is half the product of its legs: in the figure,

$$\boxed{A(\Delta ABC) = \frac{a \cdot c}{2}}.$$



Proof Let us draw a line from A parallel to BC, and let the foot of the perpendicular from C to this line be C'.



Then $ABCC'$ is a rectangle, because $AC' \parallel BC$, $m(\angle ABC) = 90°$ and $m(\angle AC'C) = 90$. So $m(\angle BAC') = 90°$.

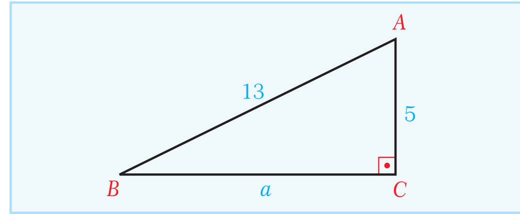
Also, ΔABC is congruent to $\Delta CC'A$ by the ASA Congruence Theorem, since $m(\angle BAC) = m(\angle ACC')$, $m(\angle ACB) = m(\angle C'AC)$ and AC is a common side.

So we can write $A(\Delta ABC) = A(\Delta CC'A) = X$.

Since $ABCC'$ is a rectangle, $A(ABCC') = a \cdot c = A(\Delta ABC) + A(\Delta CC'A) = X + X = 2X$.

So $X = A(\Delta ABC) = \frac{A(ABCC')}{2} = \frac{a \cdot c}{2}$, as required.

EXAMPLE 11 In the triangle opposite, $m(\angle C) = 90°$, $AB = 13$ cm and $AC = 5$ cm. Find $A(\Delta ABC)$.



Solution $AB = c = 13$ cm and $AC = b = 5$ cm are given, and we need to find $BC = a$ to find the area. By the Pythagorean Theorem in ΔABC,

$a^2 + 5^2 = 13^2$ so $a^2 = 169 - 25 = 144$, $a = 12$ cm.

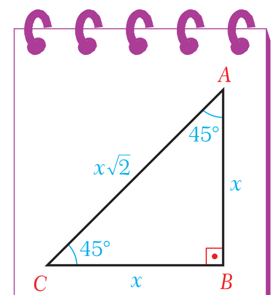
So $A(\Delta ABC) = \dfrac{a \cdot b}{2} = \dfrac{12 \cdot 5}{2} = 30 \text{ cm}^2$.

EXAMPLE 12 The hypotenuse of an isosceles right triangle measures $7\sqrt{2}$ cm. Find the area of this triangle.

Solution We know from basic trigonometry that if the sides of an isosceles right triangle measure x units then the hypotenuse measures $x\sqrt{2}$ units.

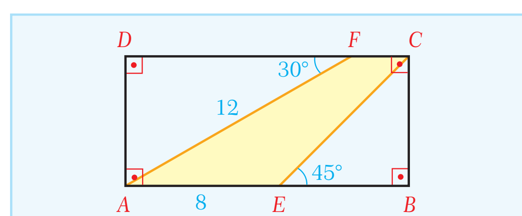
So $x\sqrt{2} = 7\sqrt{2}$, which gives $x = 7$, i.e. $a = c = 7$ cm.

So $A(\Delta ABC) = \dfrac{a \cdot c}{2} = \dfrac{7 \cdot 7}{2} = \dfrac{49}{2} \text{ cm}^2$.



an isosceles right triangle

EXAMPLE 13 In the given figure, $ABCD$ is a rectangle with $m(\angle AFD) = 30°$, $m(\angle BEC) = 45°$, $AF = 12$ and $AE = 8$. Find the area of quadrilateral $AECF$.



Solution ΔADF is a 30°-60°-90° triangle so $AD = \dfrac{12}{2} = 6$ and $DF = 6\sqrt{3}$.

Since $ABCD$ is a rectangle, $BC = AD = 6$.

ΔEBC is an isosceles right triangle (because $m(\angle BEC) = m(\angle ECB) = 45°$), so $EB = BC = 6$.

So in rectangle $ABCD$, $AD = a = 6$ and $AB = b = AE + EB = 8 + 6 = 14$.

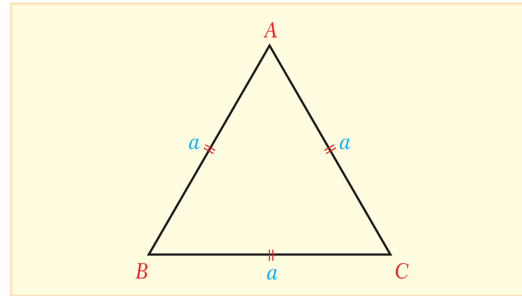
Finally, $A(AECF) = A(ABCD) - (A(\Delta ADF) + A(\Delta EBC)) = 6 \cdot 14 - (\dfrac{6 \cdot 6\sqrt{3}}{2} + \dfrac{6 \cdot 6}{2})$

$= 84 - (18\sqrt{3} + 18) = 66 - 18\sqrt{3}$.

Theorem area of an equilateral triangle

The area of an equilateral triangle with side length a is one-fourth of the product of a^2 and $\sqrt{3}$: in the figure,

$$\boxed{A(\Delta ABC) = \frac{a^2\sqrt{3}}{4}}.$$

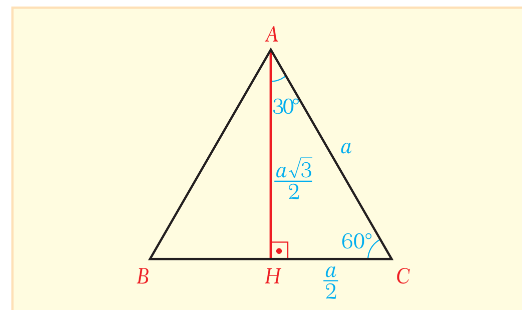


Proof Let us draw the altitude AH in ΔABC.

Since ΔABC is equilateral,

$m(\angle C) = 60°$ and so $AH = h_a = \dfrac{a\sqrt{3}}{2}$.

So $A(\Delta ABC) = \dfrac{a \cdot h_a}{2} = \dfrac{a \cdot \dfrac{a\sqrt{3}}{2}}{2} = \dfrac{a^2\sqrt{3}}{4}$.



EXAMPLE 14 One side of an equilateral triangle measures 6 cm. Find the area of this triangle.

Solution By the theorem above, $A = \dfrac{a^2\sqrt{3}}{4} = \dfrac{6^2\sqrt{3}}{4} = 9\sqrt{3}$ cm^2.

EXAMPLE 15 The height of an equilateral triangle is 10 cm. Find its area.

Solution The height is $h = \dfrac{a\sqrt{3}}{2} = 10$, so $a = \dfrac{20}{\sqrt{3}}$ cm.

So $A(\Delta ABC) = \dfrac{a^2\sqrt{3}}{4} = \dfrac{(\frac{20}{\sqrt{3}})^2\sqrt{3}}{4} = \dfrac{400\sqrt{3}}{12} = \dfrac{100\sqrt{3}}{3}$ cm^2.

EXAMPLE 16 The area and perimeter of an equilateral triangle have the same value. Find the length of one side of this triangle.

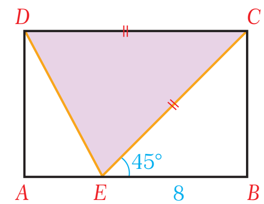
Solution Let the side measure a, then the area is $\dfrac{a^2\sqrt{3}}{4}$ and the perimeter is $3a$.

If the area and the perimeter have equal values then $\dfrac{a^2\sqrt{3}}{4} = 3a$ and so

$a = \dfrac{12}{\sqrt{3}} = 4\sqrt{3}$ is the length of one side of the triangle.

Check Yourself

1. The hypotenuse of a right triangle measures 25 units and one of its legs measures 24 units. Find the area of this triangle.
2. An isosceles right triangle has a hypotenuse of 10 units. Find its area.
3. In the figure, $ABCD$ is a rectangle, $EB = 8$, $m(\angle BEC) = 45°$ and $EC = DC$. Find the area of ΔCDE.



4. Find the area of the equilateral triangle with the given side length.

   a. 4 b. 10

5. Find the area of the equilateral triangle with the given height.

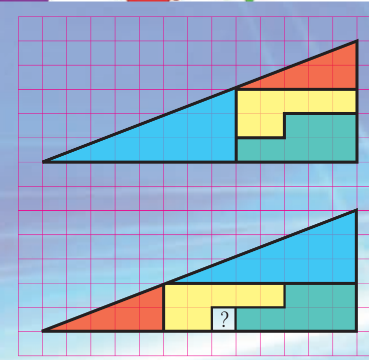
   a. $14\sqrt{3}$ b. 8 c. $2\sqrt{3}$

Answers

1. 84 2. 25 3. $32\sqrt{2}$ 4. a. $4\sqrt{3}$ b. $25\sqrt{3}$ 5. a. $196\sqrt{3}$ b. $\frac{64\sqrt{3}}{3}$ c. $4\sqrt{3}$

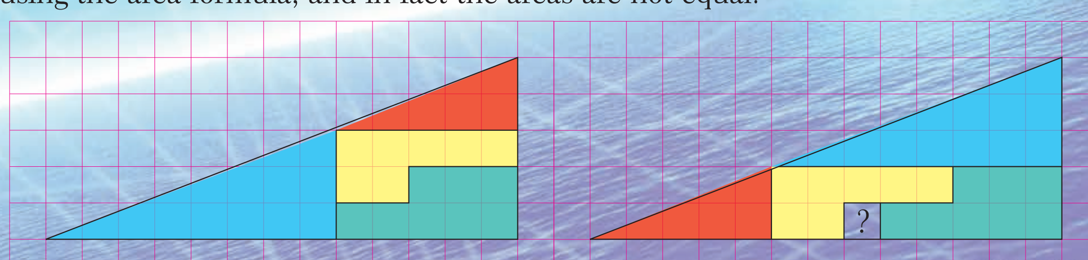
The figure shows two triangles. We cut the first triangle along the lines shown and rearrange the parts to get the second triangle.

When we do this, we can see that there is an empty space in the second triangle. Where is the missing square?



Answer

If we draw the figures accurately and with a large scale, we will see that the slopes of the red and blue triangles are different. This means that the bigger shapes are not triangles. So we cannot calculate the areas directly by using the area formula, and in fact the areas are not equal.

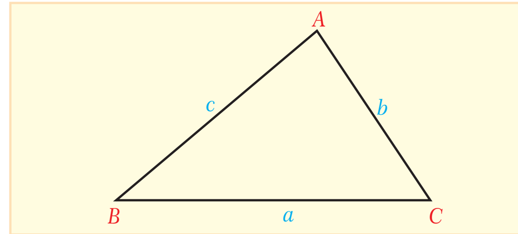


Theorem — Heron's Formula

If a triangle has sides a, b and c and perimeter $2u$ then the area of the triangle is the square root of the product of u, $u - a$, $u - b$ and $u - c$: in the figure,

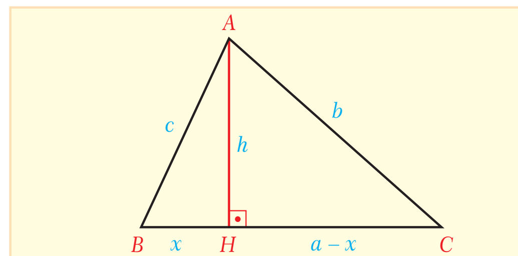
$$u = \frac{a+b+c}{2}$$

$$A(\Delta ABC) = \sqrt{u(u-a)(u-b)(u-c)}.$$



Proof

Look at the figure. Let us draw the altitude AH to BC and let $AH = h$, $BH = x$, $HC = a - x$ and $A(\Delta ABC) = A$.



By the Pythagorean Theorem in ΔABH and ΔAHC respectively we get

$x^2 + h^2 = c^2$, $h^2 = c^2 - x^2$, (1)

and $(a - x)^2 + h^2 = b^2$, $h^2 = b^2 - (a - x)^2$. (2)

From (1) and (2) we have $h^2 = c^2 - x^2 = b^2 - (a - x)^2$

$c^2 - x^2 = b^2 - (a^2 - 2ax + x^2)$ (expand the binomial)

$c^2 - \cancel{x^2} = b^2 - a^2 + 2ax - \cancel{x^2}$ (simplify)

$2ax = a^2 - b^2 + c^2$, i.e. $x = \dfrac{a^2 - b^2 + c^2}{2a}$. (3) (rearrange)

Let us use (3) in (1): $h^2 = c^2 - (\dfrac{a^2 - b^2 + c^2}{2a})^2$, i.e. $h^2 = c^2 - \dfrac{(a^2 - b^2 + c^2)^2}{4a^2}$.

Equalizing denominators gives us $h^2 = \dfrac{(2ac)^2 - (a^2 - b^2 + c^2)^2}{4a^2}$.

So $4a^2h^2 = (2ac - (a^2 - b^2 + c^2)) \cdot (2ac + a^2 - b^2 + c^2)$. (4) (difference of two squares)

We know that $A(\Delta ABC) = A = \dfrac{ah}{2}$, so $A^2 = \dfrac{a^2h^2}{4}$, i.e. $16A^2 = 4a^2h^2$. So

$16A^2 = (b^2 - (a^2 - 2ac + c^2)) \cdot (a^2 + 2ac + c^2 - b^2)$ (from (4))

$16A^2 = (b^2 - (a - c)^2) \cdot ((a + c)^2 - b^2)$ (use $a^2 \pm 2ac + c^2 = (a \pm c)^2$)

$16A^2 = (b - a + c)(b + a - c)(a + c - b)(a + c + b)$. (5) (difference of two squares)

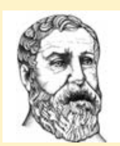
We know that the perimeter of ΔABC is $2u$, i.e. $a + b + c = 2u$.

Let us substitute this in (5):

$16A^2 = (2u - 2a)(2u - 2c)(2u - 2b)(2u)$, i.e. $A^2 = \dfrac{2u \cdot 2(u-a) \cdot 2(u-b) \cdot 2(u-c)}{16}$

and so $A = \sqrt{u \cdot (u-a) \cdot (u-b) \cdot (u-c)}$, as required.

HERON OF ALEXANDRIA (10-75 AD)



Heron of Alexandria was a Greek mathematician and inventor who lived in Alexandria in Egypt. Heron was mainly interested in the practical study of mechanics and engineering. He also studied geometry, optics, astronomy and architecture. In geometry, he found the area of a triangle by using square roots and the lengths of the sides. He invented many machines and devices such as fountains and syphons, and he also invented the first steam powered device.

Heron wrote around fifteen books about mathematics, engineering and astronomy, including *Katoptirikos* (about optics), *Automata*, *Mechanica*, *Geometrica* and *Stereometrica*. The proof of the formula presented here appears in his book *Metrica*.

EXAMPLE 17 Find the area of the triangle with side lengths 4 cm, 5 cm and 7 cm.

Solution We can use Heron's Formula with $u = \dfrac{a+b+c}{2} = \dfrac{4+5+7}{2} = 8$:

$$A(\Delta ABC) = \sqrt{8(8-4)(8-5)(8-7)} = \sqrt{8 \cdot 4 \cdot 3 \cdot 1} = 4\sqrt{6}\ \text{cm}^2.$$

EXAMPLE 18 The sides of a triangle ABC are $a = 7$, $b = 9$ and $c = 12$. Find h_c.

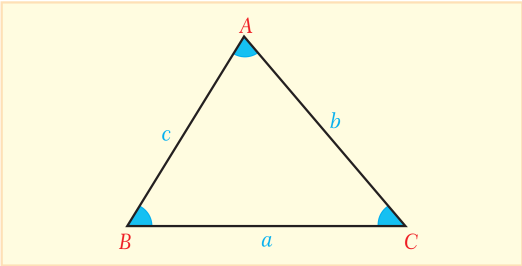
Solution By Herons's Formula with $u = \dfrac{7+9+12}{2} = 14$ we have

$$A(\Delta ABC) = \sqrt{14 \cdot (14-7) \cdot (14-9) \cdot (14-12)} = 14\sqrt{5}.$$

Also, $A(\Delta ABC) = \dfrac{c \cdot h_c}{2}$. So $14\sqrt{5} = \dfrac{12 \cdot h_c}{2}$ and $h_c = \dfrac{7\sqrt{5}}{3}$.

Theorem trigonometric formula for the area of a triangle

The area of a triangle is half the product of any two sides and the sine of the angle between these two sides: in the figure,



$$\boxed{\begin{aligned} A(\Delta ABC) &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}ac\sin B \\ &= \frac{1}{2}bc\sin A \end{aligned}}.$$

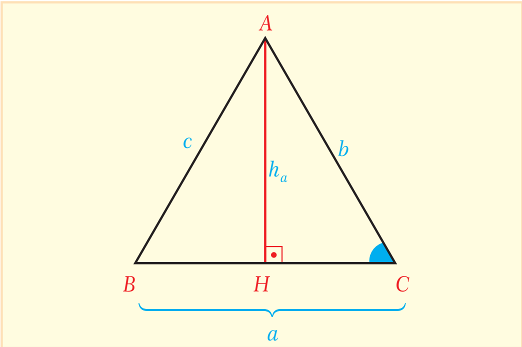
Proof Look at the figure.

Let us draw the altitude from the vertex A to side BC, so $AH = h_a$. From the figure,

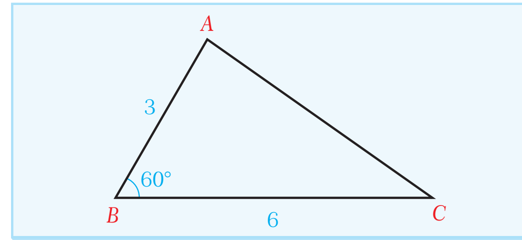
$\sin C = \dfrac{h_a}{b}$, i.e. $h_a = b \cdot \sin C$.

So $A(\Delta ABC) = \dfrac{a \cdot h_a}{2} = \dfrac{1}{2}ab \cdot \sin C$.

The proofs for the other pairs of sides are similar.



EXAMPLE 19 In the figure, $AB = c = 3$ cm, $BC = a = 6$ cm and $m(\angle B) = 60°$.
What is $A(\Delta ABC)$?



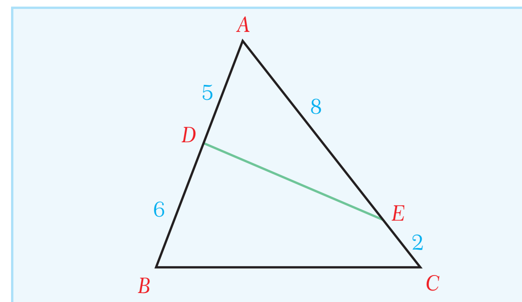
Solution We know that $a = 6$, $c = 3$ and $m(\angle B) = 60°$.

Since $A(\Delta ABC) = \frac{1}{2} ac \cdot \sin B$, we have

$$A(\Delta ABC) = \frac{1}{2} \cdot 6 \cdot 3 \cdot \sin 60° = \frac{9\sqrt{3}}{2} \text{ cm}^2.$$

EXAMPLE 20 In the figure, $AD = 5$ cm,
$BD = 6$ cm,
$AE = 8$ cm and
$EC = 2$ cm.

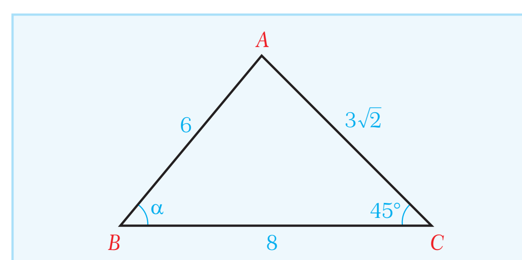
What is $\frac{A(\Delta ADE)}{A(\Delta ABC)}$?



Solution We can find both of the areas using the trigonometric formulas we have just seen.
We know $AD = 5$, $AE = 8$, $AB = 11$ and $AE = 10$, so

$$\frac{A(\Delta ADE)}{A(\Delta ABC)} = \frac{\frac{1}{2} \cdot AD \cdot AE \cdot \sin A}{\frac{1}{2} \cdot AB \cdot AC \cdot \sin A} = \frac{AD \cdot AE}{AB \cdot AC} = \frac{5 \cdot 8}{11 \cdot 10} = \frac{4}{11}.$$

EXAMPLE 21 In the figure,
$AB = 6$ cm,
$BC = 8$ cm,
$AC = 3\sqrt{2}$ cm and
$m(\angle C) = 45°$.
Find α.



Solution We have $AB = c = 6$, $BC = a = 8$ and $AC = b = 3\sqrt{2}$, and we know $\sin 45° = \frac{\sqrt{2}}{2}$.

Since $A(\Delta ABC) = \frac{1}{2} ab \cdot \sin C = \frac{1}{2} ac \cdot \sin \alpha$, we have $3\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 6 \cdot \sin \alpha$ which gives us

$\sin \alpha = \frac{1}{2}$, i.e. $\alpha = 30°$ or $\alpha = 150°$.

Since $m(\angle C) = 45°$, $\alpha = 150°$ is impossible, and so $\alpha = 30°$.

EXAMPLE 22 The sides of ΔPQR are $p = 5$, $q = 12$ and $r = 15$. Find $\sin R$.

Solution Let us find $A(\Delta PQR)$ by using both Heron's Formula and the trigonometric formula for the area of a triangle.

By Heron's Formula with $u = \dfrac{5+12+15}{2} = 16$,

$A(\Delta PQR) = \sqrt{16 \cdot (16-5) \cdot (16-12) \cdot (16-15)} = 8\sqrt{11}$.

By the trigonometric formula, $A(\Delta PQR) = \dfrac{1}{2} \cdot p \cdot q \cdot \sin R = \dfrac{1}{2} \cdot 5 \cdot 12 \cdot \sin R = 30 \cdot \sin R$.

So $30 \cdot \sin R = 8\sqrt{11}$, and so $\sin R = \dfrac{8\sqrt{11}}{30} = \dfrac{4\sqrt{11}}{15}$.

Check Yourself

1. A triangle has sides of 13 cm, 14 cm and 15 cm. Find its area.
2. The sides of a triangle are $a = 3$, $b = 5$ and $c = 6$. Find the three heights of this triangle.
3. $AB = 8$ cm and $AC = 6\sqrt{3}$ cm are two sides of a triangle ABC. Find $A(\Delta ABC)$ if
   a. $m(\angle A) = 30°$. b. $m(\angle A) = 60°$. c. $m(\angle A) = 90°$.
4. The sides of a triangle are $a = 8$, $b = 7$ and $c = 9$. What is the sine of angle B?

Answers

1. 84 cm^2 2. $h_a = \dfrac{4\sqrt{14}}{3}$, $h_b = \dfrac{4\sqrt{14}}{5}$, $h_c = \dfrac{2\sqrt{14}}{3}$ 3. a. $12\sqrt{3}\text{ cm}^2$ b. 36 cm^2 c. $24\sqrt{3}\text{ cm}^2$ 4. $\dfrac{\sqrt{5}}{3}$

Definition

incircle, inscribed circle, incenter, inradius

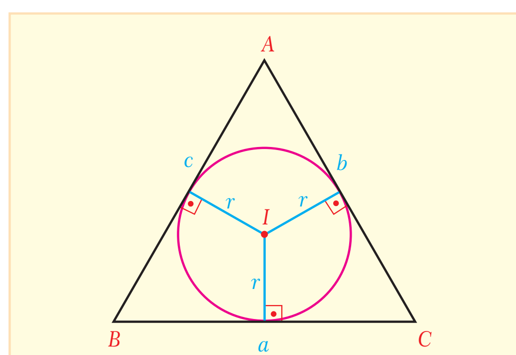
The **incircle** (or **inscribed circle**) of a triangle is a circle inside the triangle which is tangent to each of its sides. The center of the incircle is called the **incenter**. It lies at the intersection point of the angle bisectors of the triangle. The radius of the incircle is called the **inradius** (r).

Theorem

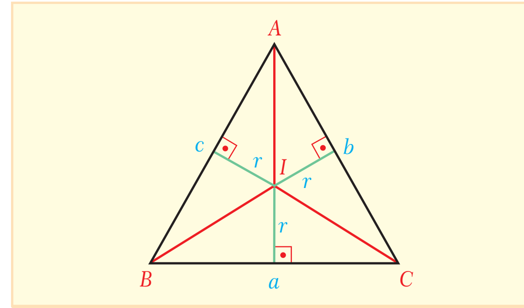
area of a triangle by its inradius

The area of a triangle is the product of half its perimeter and its inradius: in the figure, if $a + b + c = 2u$ then

$$\boxed{A(\Delta ABC) = u \cdot r}.$$



Proof Look at the figure. Let I be the incenter of ΔABC and let r be its inradius. If we connect I and the vertices A, B and C we can write



$$A(\Delta ABC) = A(\Delta AIB) + A(\Delta BIC) + A(\Delta AIC)$$

$$= \frac{c \cdot r}{2} + \frac{a \cdot r}{2} + \frac{b \cdot r}{2}$$

$$= \frac{a+b+c}{2} \cdot r$$

$$= u \cdot r.$$

EXAMPLE 23 A triangle with perimeter 20 cm has inradius 6 cm. Find the area of this triangle.

Solution If the perimeter is $2u$ then $2u = 20$, i.e. $u = 10$.

By the theorem we have just seen, $A = u \cdot r = 10 \cdot 6 = 60$ cm^2.

EXAMPLE 24 The sides of a triangle measure 7, 8 and 9 units. Find the radius of the incircle of this triangle.

Solution $u = \dfrac{7+8+9}{2} = 12$

$$A = \sqrt{u(u-a)(u-b)(u-c)} = \sqrt{12(12-7)(12-8)(12-9)} \qquad \text{(Heron's Formula)}$$

$$= \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = 12\sqrt{5}$$

Also, $A = u \cdot r = 12 \cdot r = 12\sqrt{5}$, so $r = \sqrt{5}$.

Definition

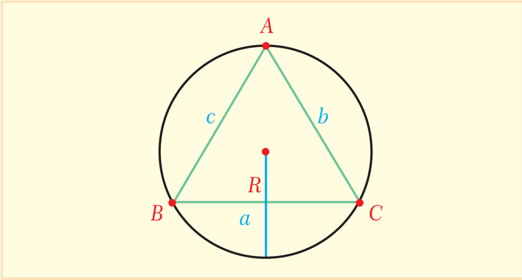
circumcircle, circumscribed circle, circumradius

The circle which passes through all three vertices of a triangle is called the **circumcircle** (or **circumscribed circle**) of the triangle. The radius of the circumcircle is called the **circumradius** (R).

Theorem area of a triangle by its circumradius

The area of a triangle is equal to the ratio of the product of the sides to four times its circumradius: in the figure,

$$\boxed{A(\Delta ABC)=\frac{a\cdot b\cdot c}{4R}}\,.$$



Proof The law of sines tells us that $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$.

So $a = 2R \cdot \sin A$, i.e. $\sin A=\dfrac{a}{2R}$.

Substituting this in $A(\Delta ABC)=\dfrac{1}{2}\cdot b\cdot c\cdot\sin A$ gives us $A(\Delta ABC)=\dfrac{1}{2}\cdot b\cdot c\cdot\dfrac{a}{2R}=\dfrac{a\cdot b\cdot c}{4R}$, as required.

EXAMPLE 25 The sides of a triangle measure 8 cm, 10 cm and 12 cm. Find the circumradius R of this triangle.

Solution Let $u=\dfrac{8+10+12}{2}=15$, then by Heron's Formula the area of the triangle will be

$A=\sqrt{15\cdot(15-8)\cdot(15-10)\cdot(15-12)}=\sqrt{15\cdot7\cdot5\cdot3}=15\sqrt{7}\text{ cm}^2$.

Using the formula $A(\Delta ABC)=\dfrac{a\cdot b\cdot c}{4R}$ gives us $15\sqrt{7}=\dfrac{8\cdot10\cdot12}{4R}$, i.e. $R=\dfrac{16\sqrt{7}}{7}$ cm.

EXAMPLE 26 The sides of a right triangle measure 6 cm and 8 cm. Find the sum of the circumradius and the inradius of this triangle.

Solution From geometry we know that if the vertices of a right triangle all lie on the same circle then the hypotenuse is the diameter of this circle. So the hypotenuse is the diameter of the circumcircle ($2R$). By the Pythagorean Theorem, $6^2 + 8^2 = (2R)^2$. So $2R = 10$, i.e. $R = 5$ cm.

Also, if the triangle is a right triangle then its area is $A=\dfrac{a\cdot c}{2}=\dfrac{6\cdot8}{2}=24\text{ cm}^2$.

Let $u=\dfrac{a+b+c}{2}=\dfrac{6+8+10}{2}=12$, then since $A = u \cdot r$ we have $24 = 12 \cdot r$, i.e. $r = 2$ cm.

So the sum of the circumradius and inradius is $R + r = 5 + 2 = 7$ cm.

EXAMPLE 27 A circle has radius 8 cm. Find the area of the equilateral triangle whose vertices lie on this circle.

Solution We can write the area of the equilateral triangle in two ways:

$A = \dfrac{a^2\sqrt{3}}{4}$ and $A = \dfrac{a \cdot a \cdot a}{4 \cdot R}$.

If we equate these expressions using $R = 8$ cm, we get $\dfrac{a^2\sqrt{3}}{4} = \dfrac{a \cdot a \cdot a}{4 \cdot 8}$ and $a = 8\sqrt{3}$.

So $A = \dfrac{a^2\sqrt{3}}{4} = \dfrac{(8\sqrt{3})^2 \cdot \sqrt{3}}{4} = \dfrac{192\sqrt{3}}{4} = 48\sqrt{3}$ cm^2.

Check Yourself

1. The legs of a right triangle measure 5 cm and 12 cm. Find the inradius of this triangle.
2. In ΔABC, $m(\angle A) = 90°$ and $AB = 3\sqrt{2}$. If $m(\angle C) = 45°$, find the circumradius of ΔABC.
3. The sides of a triangle measure 12, 9 and 7 units. Find the lengths R and r of the circumradius and inradius of this triangle.
4. One of the legs of a right triangle measures 9 units and the diameter of its circumcircle is 15. Find the inradius of this triangle.
5. An equilateral triangle has side length 6. Find the sum of the inradius and circumradius of this triangle.
6. The base of an isosceles triangle measures 24 units and the other sides are each 13 units long. Find the inradius r and circumradius R of this triangle.

Answers

1. 2 cm 2. 3 3. $R = \dfrac{27\sqrt{5}}{10}$, $r = \sqrt{5}$ 4. 3 5. $3\sqrt{3}$ 6. $r = \dfrac{12}{5}$, $R = \dfrac{169}{10}$

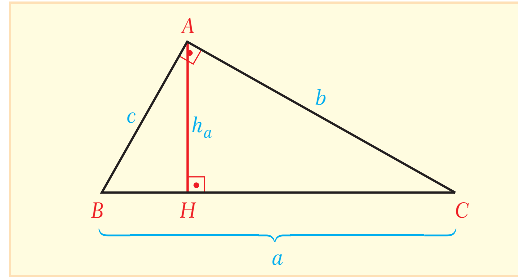
C. PROPERTIES OF THE AREA OF A TRIANGLE

So far we have learned several formulas for the area of a triangle. However, sometimes these formulas may not be enough to solve a problem, or using them can make the solution longer. In this section we look at some properties of triangles that can help us to solve problems more directly.

Property 1

In a right triangle, the product of the legs is equal to the product of the hypotenuse and the length of the altitude to the hypotenuse: in the figure,

$$\boxed{a \cdot h_a = b \cdot c}\ .$$



Proof We can write the area of a right triangle in two ways: $A(\Delta ABC) = \dfrac{a \cdot h_a}{2}$ and $A(\Delta ABC) = \dfrac{b \cdot c}{2}$. Equating and simplifying gives us $a \cdot h_a = b \cdot c$.

EXAMPLE 28 The legs of a right triangle measure 7 cm and 24 cm. Find the height drawn to the hypotenuse of this triangle.

Solution Let the hypotenuse be a. By the Pythagorean Theorem we have

$a^2 = 7^2 + 24^2 = 49 + 576 = 625$, i.e. $a = 25$ cm.

Now using $a \cdot h_a = b \cdot c$ gives us $25 \cdot h_a = 7 \cdot 24$, i.e. $h_a = \dfrac{168}{25}$ cm.

As an exercise, try solving this problem using only the formulas we studied in the previous section. Can you do it?

Property 2

If the base lengths and heights of two triangles are the same then their areas are equal.

Proof Let the area of the first triangle be $A_1 = \dfrac{base_1 \cdot height_1}{2}$, and the second area be

$A_2 = \dfrac{base_1 \cdot height_1}{2}$ (because the bases and heights are equal).

Then $A_1 = A_2$, as required.

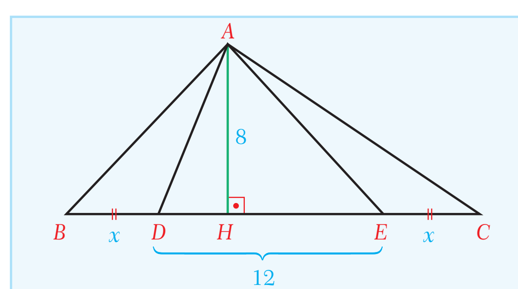
EXAMPLE 29 In the figure,

$A(\Delta ABC) = 112$ cm^2,

$DE = 12$ cm and

$AH = 8$ cm are given.

Find $BD = EC = x$.



Solution In the figure, $AH = h_a = 8$ cm is the common height of ΔABD, ΔADE and ΔAEC. The base lengths $BD = EC = x$ are the same and their heights are also equal, so by Property 2 we have

$A(\Delta ABD) = A(\Delta AEC) = S$.

Also, $A(\Delta ADE) = \dfrac{DE \cdot AH}{2} = \dfrac{12 \cdot 8}{2} = 48 \text{ cm}^2$.

From the figure we can say

$A(\Delta ABC) = A(\Delta ABD) + A(\Delta ADE) + A(\Delta AEC)$

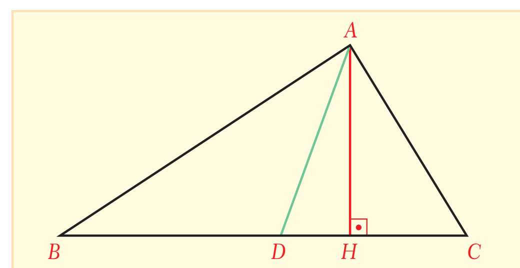
$112 = S + 48 + S$

$2S = 64,\ S = 32 \text{ cm}^2$.

Now, since $A(\Delta ABD) = S = 32 = \dfrac{x \cdot h_a}{2}$, we have $64 = 8 \cdot x$, i.e. $x = 8$ cm.

Property 3

A median of a triangle divides the area of the triangle into two equal parts: in the figure, if $BD = DC$ then $A(\Delta BAD) = A(\Delta DAC)$.



Proof Let AD be the median and AH be the altitude, as in the figure. So $BD = DC$.

We can say that AH is the altitude of both of the triangles ΔABD and ΔADC, so

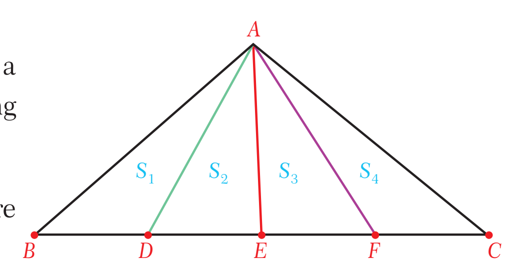
$A(\Delta ABD) = \dfrac{BD \cdot AH}{2}$ and $A(\Delta ADC) = \dfrac{DC \cdot AH}{2}$.

But since $BD = DC$ we can write $\dfrac{BD \cdot AH}{2} = \dfrac{DC \cdot AH}{2}$, which means $A(\Delta ABD) = A(\Delta ADC)$.

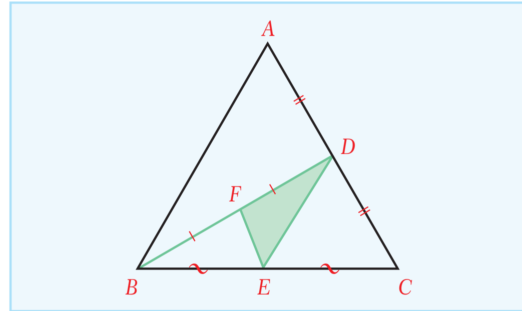
Conclusion

By applying Property 3 repeatedly, we can see that a triangle can be divided into equal parts by taking successive medians.

For example, if $BD = DE = EF = FC$ in the figure opposite then $S_1 = S_2 = S_3 = S_4$.



EXAMPLE 30 The figure shows a triangle ABC. BD is the median of side AC in ΔABC, DE is the median of side BC in ΔBCD, and EF is the median of side BD in ΔBDE. Given that $A(\Delta DEF) = 5\ \text{cm}^2$, find $A(\Delta ABC)$.



Solution In ΔBDE, $BF = FD$ so

$A(\Delta BEF) = A(\Delta DEF) = 5\ \text{cm}^2$ and so $A(\Delta BDE) = 10\ \text{cm}^2$.

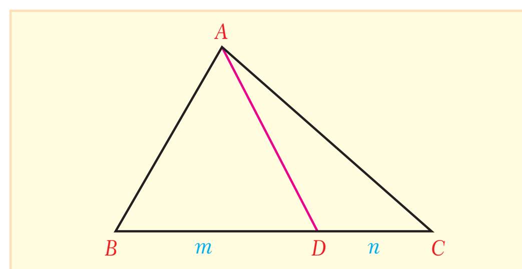
In ΔBCD, $BE = EC$ so $A(\Delta CDE) = A(\Delta BDE) = 10\ \text{cm}^2$ and so $A(\Delta BCD) = 20\ \text{cm}^2$.

In ΔABC, $AD = DC$ so $A(\Delta ABD) = A(\Delta BCD) = 20\ \text{cm}^2$ and so $A(\Delta ABC) = 40\ \text{cm}^2$.

Property 4

If the heights of two triangles are the same then the ratio of their areas is the same as the ratio of their base lengths: in the figure,

$$\boxed{\frac{A(\Delta ABD)}{A(\Delta ADC)} = \frac{m}{n}}.$$



Proof Triangles ΔABD and ΔADC in the figure both have common altitude $AH = h_a$.

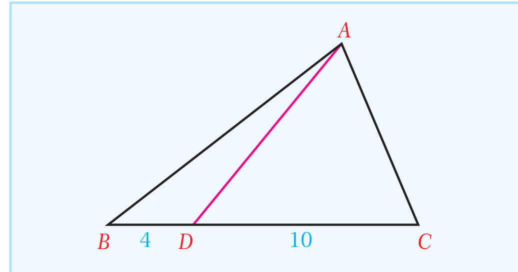
So $\dfrac{A(\Delta ABD)}{A(\Delta ADC)} = \dfrac{\dfrac{AH \cdot BD}{2}}{\dfrac{AH \cdot DC}{2}}$.

Canceling common terms gives us $\dfrac{A(\Delta ABD)}{A(\Delta ADC)} = \dfrac{BD}{DC} = \dfrac{m}{n}$.

Note

We can use the letter S to mean the common multiplier in triangle ratio problems which use this property: $A(\Delta ABD) = mS$ and $A(\Delta ADC) = nS$.

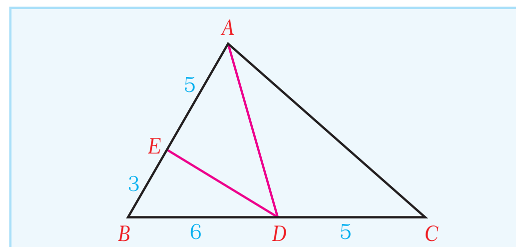
EXAMPLE 31 In the figure, $A(\Delta ABC) = 70$ cm^2, $BD = 4$ cm and $DC = 10$ cm are given. Find $A(\Delta ABD)$.



Solution By Property 4 we can write $A(\Delta ABD) = 4S$ and $A(\Delta ADC) = 10S$. Since $A(\Delta ABC) = A(\Delta ABD) + A(\Delta ADC)$, we have $70 = 4S + 10S$, i.e. $14S = 70$ and so $S = 5$ cm^2.

So $A(\Delta ABD) = 4S = 4 \cdot 5 = 20$ cm^2.

EXAMPLE 32 In the figure, $AE = 5$ cm, $EB = 3$ cm, $BD = 6$ cm and $DC = 5$ cm. The area of ΔAED is 15 cm^2. What is the area of ΔABC?



Solution In ΔABD, $AE = 5$ and $EB = 3$ so we can write $A(\Delta AED) = 5S$ and $A(\Delta EBD) = 3S$.

So $A(\Delta AED) = 5S = 15$, i.e. $S = 3$ cm^2.

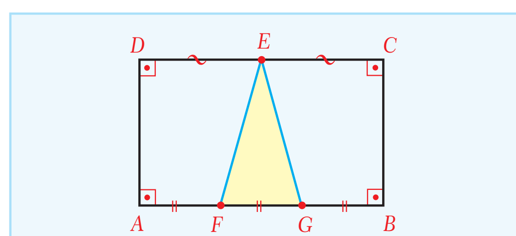
So $A(\Delta ABD) = A(\Delta AED) + A(\Delta EBD) = 5S + 3S = 8S = 8 \cdot 3 = 24$ cm^2.

Since $BD = 6$ and $DC = 5$ we can write $A(\Delta ABD) = 6X$ and $A(\Delta ADC) = 5X$.

So $A(\Delta ABD) = 6X = 24$, i.e. $X = 4$ cm^2.

So $A(\Delta ABC) = A(\Delta ABD) + A(\Delta ADC) = 6X + 5X = 11X = 11 \cdot 4 = 44$ cm^2.

EXAMPLE 33 In the figure, $ABCD$ is a rectangle. $DE = EC$, $AF = FG = GB$ and $A(\Delta EFG) = 6$ cm^2 are given. Find $A(ABCD)$.



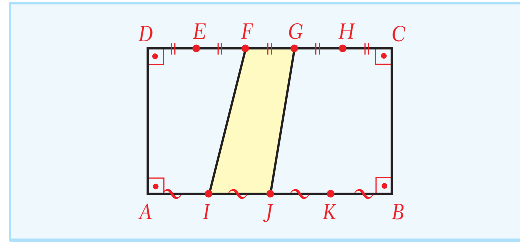
Solution Let us draw EA and EB. By Property 4, $A(\Delta AEF) = A(\Delta EFG) = A(\Delta EGB) = 6$ cm^2.

So $A(\Delta EAB) = 6 + 6 + 6 = 18$ cm^2.

We also know that $A(\Delta EAB) = \dfrac{A(ABCD)}{2}$ by the properties of a rectangle.

So $A(ABCD) = 2 \cdot A(\Delta EAB) = 2 \cdot 18 = 36$ cm^2.

EXAMPLE 34 In the figure, $ABCD$ is a rectangle. DC is divided into five equal parts and AB is divided into four equal parts. Given that $A(ABCD) = 180 \text{ cm}^2$, find $A(IJGF)$.



Solution Let us divide $IJGF$ into two triangles and find the area of each triangle. We know that if we connect any point on one side of a rectangle to the two non-adjacent vertices then the area of the triangle formed will be half the area of the rectangle. Let us draw the lines GI, DI and CI.

By the properties of a rectangle we have $A(\Delta DIC) = \dfrac{A(ABCD)}{2} = \dfrac{180}{2} = 90 \text{ cm}^2$.

The base DC of ΔDIC is divided into five equal parts, so by Property 4 we can write $A(\Delta DIC) = 5S = 90 \text{ cm}^2$.

So $A(\Delta IFG) = S = \dfrac{90}{5} = 18 \text{ cm}^2$.

In the same way we can draw GA and GB to get

$A(\Delta ABG) = \dfrac{A(ABCD)}{2} = \dfrac{180}{2} = 90 \text{ cm}^2$.

Since AB is divided into four equal parts, we have

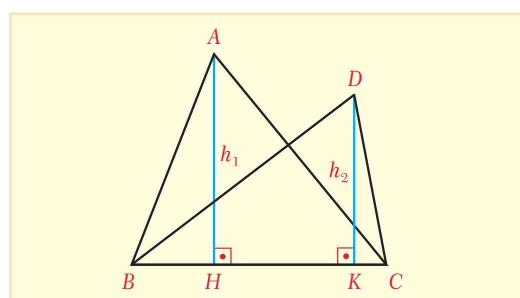
$A(\Delta IGJ) = \dfrac{90}{4} = \dfrac{45}{2} \text{ cm}^2$.

Finally, $A(IJGF) = A(\Delta IFG) + A(\Delta IGJ) = 18 + \dfrac{45}{2} = \dfrac{81}{2} \text{ cm}^2$.

Property 5

If the bases of two triangles are the same then the ratio of their areas is the same as the ratio of their heights: in the figure,

$$\boxed{\frac{A(\Delta ABC)}{A(\Delta DBC)} = \frac{h_1}{h_2}}.$$

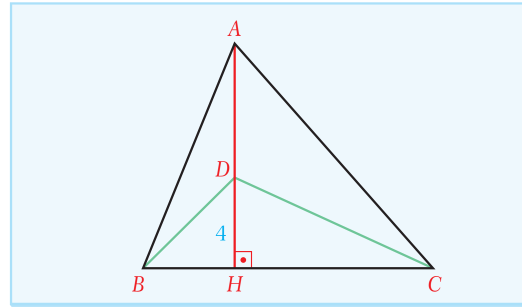


Proof BC is a common base, so

$$\frac{A(\Delta ABC)}{A(\Delta DBC)} = \frac{\dfrac{BC \cdot h_1}{2}}{\dfrac{BC \cdot h_2}{2}}.$$

Canceling common terms gives us $\dfrac{A(\Delta ABC)}{A(\Delta DBC)} = \dfrac{h_1}{h_2}$, as required.

EXAMPLE 35 In the figure, $BC = 8$ cm and $DH = 4$ cm.
Given that $A(ABDC) = 20\text{ cm}^2$, find AD.



Solution Let $AD = x$, then $AH = x + 4$.

We have $A(\Delta ABC) = A(ABDC) + A(\Delta DBC)$.

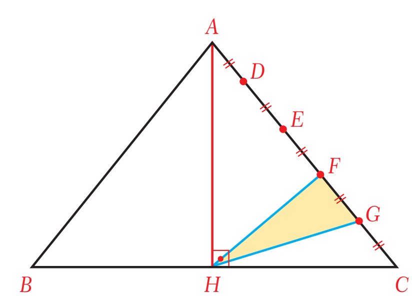
By Property 5, $\dfrac{A(\Delta ABC)}{A(\Delta DBC)} = \dfrac{AH}{DH}$, so

$$\frac{20+\dfrac{4\cdot 8}{2}}{\dfrac{4\cdot 8}{2}} = \frac{x+4}{4},$$

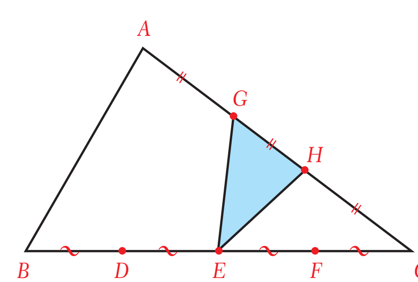
$\dfrac{36}{16} = \dfrac{x+4}{4}$, i.e. $\dfrac{9}{4} = \dfrac{x+4}{4}$, $x + 4 = 9$, $x = 5$ cm. So $AD = 5$ cm.

Check Yourself

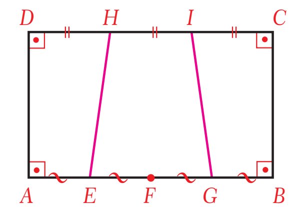
1. The legs of a right triangle measure 5 cm and 12 cm. Find the length of the altitude to the hypotenuse of this triangle.
2. Two parallel lines are given. Two points A and B lie on one of the lines and points C, D and E lie on the other line. What can you say about $A(\Delta ABC)$, $A(\Delta ABD)$ and $A(\Delta ABE)$?
3. In the figure, ΔABC is an isosceles triangle with $AB = AC$. Given that $AD = DE = EF = FG = GC$ and AH is the altitude to side BC, find $\dfrac{A(\Delta FGH)}{A(\Delta ABC)}$.



4. In the figure, BC is divided into four equal parts and AC is divided into three equal parts.
   If $A(\Delta EGH) = 4\text{ cm}^2$, find $A(\Delta ABC)$.



5. In the figure, $ABCD$ is a rectangle. AB is divided into four equal parts and DC is divided into three equal parts. If $A(ABCD) = 120 \text{ cm}^2$, find $A(EGIH)$.



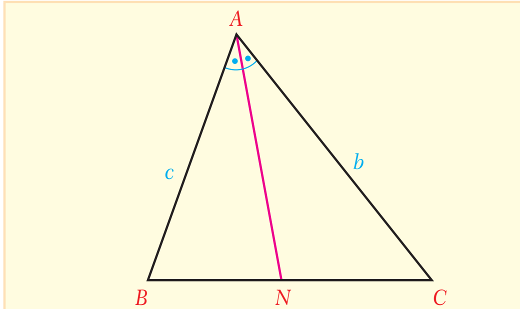
Answers

1. $\dfrac{60}{13}$ cm 2. they are equal 3. $\dfrac{1}{10}$ 4. 24 cm^2 5. 50 cm^2

Property 6

The ratio of the areas of the two triangles formed by the bisector of an angle in a triangle is the same as the ratio of the lengths of the two sides separated by the bisector: in the figure,

$$\boxed{\frac{A(\Delta ABN)}{A(\Delta ANC)} = \frac{c}{b}}.$$



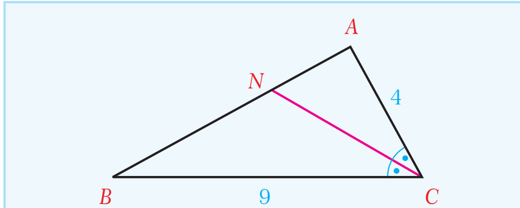
Proof In the figure, let us draw perpendiculars from the point N to the sides AB and AC and let the intersection points of these perpendiculars and the sides be D and E respectively.

Since a point on the bisector of an angle is the same distance from the two sides of the angle, we can write $ND = NE = x$.

So $\dfrac{A(\Delta ABN)}{A(\Delta ANC)} = \dfrac{\frac{AB \cdot ND}{2}}{\frac{AC \cdot NE}{2}} = \dfrac{AB \cdot x}{AC \cdot x} = \dfrac{AB}{AC} = \dfrac{c}{b}$, as required.

EXAMPLE 36

In the figure, CN is the bisector of $\angle C$. Given that $AC = 4$ cm, $BC = 9$ cm and $A(\Delta ANC) = 12 \text{ cm}^2$, find $A(\Delta ABC)$.



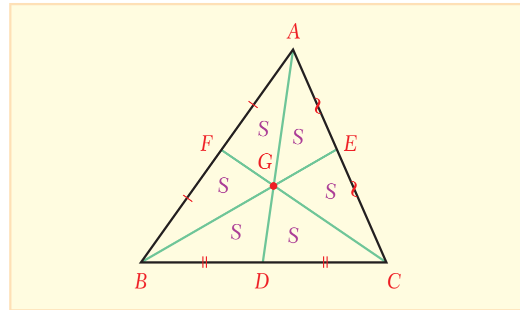
Solution By Property 6 we have $\dfrac{A(\Delta ANC)}{A(\Delta BNC)} = \dfrac{AC}{BC}$, so

$\dfrac{12}{A(\Delta BNC)} = \dfrac{4}{9}$ and so $A(\Delta BNC) = 27 \text{ cm}^2$.

So $A(\Delta ABC) = A(\Delta ANC) + A(\Delta BNC) = 12 + 27 = 39 \text{ cm}^2$.

Property 7

The medians of a triangle together divide the area of the triangle into six equal parts.



Proof Look at the figure. Let D, E and F be endpoints of the medians to sides BC, AC and AB, respectively. So G is the centroid of ΔABC.

If AD is a median then by the properties of a centroid we can write $GD = x$ and $GA = 2x$.

Then by Property 4 we have

$A(\Delta BDG) = S$ and $A(\Delta BGA) = 2S$.

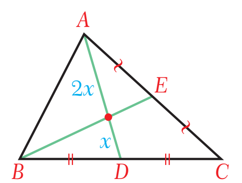
In ΔBGA, GF is a median so

$A(\Delta AFG) = A(\Delta BFG) = S$.

So we have

$A(\Delta AFG) = A(\Delta BFG) = A(\Delta BDG) = S$. (1)

The centroid of a triangle divides each median in *BGA*, the ratio 1:2:



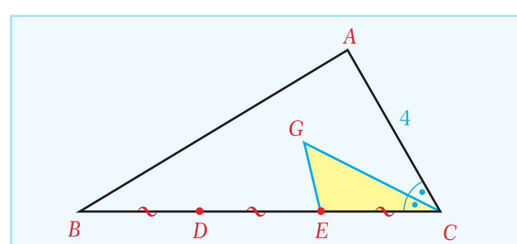
Since AD is a median, we have $A(\Delta ABD) = A(\Delta ADC)$.

By similar reasoning to the above we can get

$A(\Delta AGE) = A(\Delta EGC) = A(\Delta GDC) = S$. (2)

Combining (1) and (2) shows that all six areas are equal to each other.

EXAMPLE 37 In the figure, G is the centroid of ΔABC. Given that $BD = DE = EC$ and $A(\Delta GEC) = 8\text{ cm}^2$, find $A(\Delta ABC)$.



Solution By Property 4,

$A(\Delta GBD) = A(\Delta GDE) = A(\Delta GEC) = 8\text{ cm}^2$.

So $A(\Delta BGC) = A(\Delta GBD) + A(\Delta GDE) + A(\Delta GEC) = 8 + 8 + 8 = 24\text{ cm}^2$.

By Property 7, $A(\Delta BGC) = 2S = 24$ so $S = 12$ and $A(\Delta ABC) = 6S$.

So $A(\Delta ABC) = 6S = 6 \cdot 12 = 72\text{ cm}^2$.

Property 8

Let ABC be a triangle with inradius r and altitudes, h_a, h_b and h_c. Then

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

Proof We know $A(\Delta ABC) = A = \frac{a \cdot h_a}{2} = \frac{b \cdot h_b}{2} = \frac{c \cdot h_c}{2}$, so we have $h_a = \frac{2A}{a}$, $h_b = \frac{2A}{b}$ and $h_c = \frac{2A}{c}$.

Rearranging these gives us $\frac{1}{h_a} = \frac{a}{2A}$, $\frac{1}{h_b} = \frac{b}{2A}$ and $\frac{1}{h_c} = \frac{c}{2A}$.

If we add these terms we have

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a}{2A} + \frac{b}{2A} + \frac{c}{2A} = \frac{a+b+c}{2A} = \frac{2u}{2A} = \frac{u}{A}.$$

Since $A = u \cdot r$ we can write $\frac{u}{A} = \frac{u}{u \cdot r} = \frac{1}{r}$. So $\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$, as required.

EXAMPLE 38 The lengths of the altitudes of a triangle are 4 cm, 6 cm and 8 cm. What is the inradius of this triangle?

Solution By Property 8 we have $\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$.

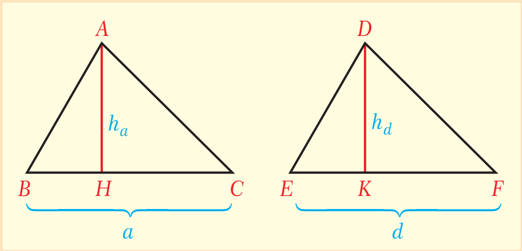
So $\frac{1}{r} = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{6+4+3}{24} = \frac{13}{24}$, i.e. $r = \frac{24}{13}$ cm.

Property 9

If two triangles are similar then the ratio of their areas is equal to the square of the ratio of similarity.

Proof Look at the figure. Let $\Delta ABC \sim \Delta DEF$.

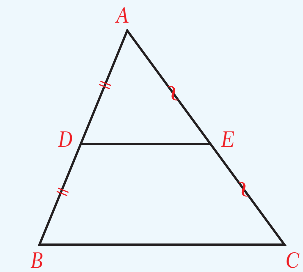
Then $\frac{a}{d} = k$ and $\frac{h_a}{h_d} = k$, where k is the ratio of similarity.



So $$\frac{A(\Delta ABC)}{A(\Delta DEF)} = \frac{\frac{a \cdot h_a}{2}}{\frac{d \cdot h_d}{2}} = \frac{a \cdot h_a}{d \cdot h_d} = \frac{a}{d} \cdot \frac{h_a}{h_d}$$

$$= k \cdot k = k^2, \text{ as required.}$$

EXAMPLE 39 In the triangle opposite, D and E are the midpoints of sides AB and AC respectively. Find $\dfrac{A(\Delta ADE)}{A(\Delta ABC)}$.



Solution If D and E are the midpoints of the sides then $DE \parallel BC$ and so $\Delta ADE \sim \Delta ABC$.

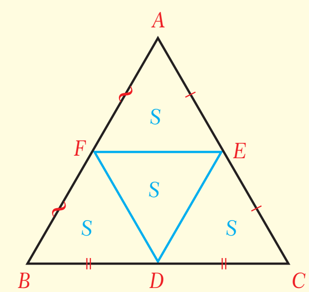
The ratio of similarity is $k = \dfrac{AD}{AB} = \dfrac{1}{2}$. ($D$ is the midpoint of AB)

So by Property 9, $\dfrac{A(\Delta ADE)}{A(\Delta ABC)} = k^2 = (\dfrac{1}{2})^2 = \dfrac{1}{4}$.

Rule

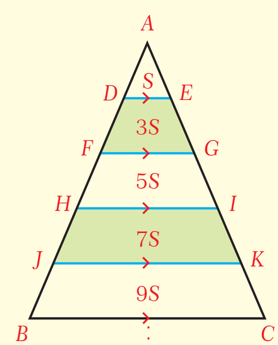
When we connect the midpoints of the sides of a triangle, the area of the triangle is divided into four equal parts: in the figure,

$A(\Delta AFE) = A(\Delta BDF) = A(\Delta DEC) = A(\Delta DEF) = S$.



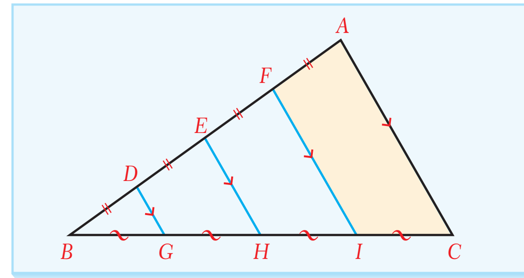
Rule

If we divide two sides of a triangle into equal lengths and connect the dividing points with parallel lines, the areas of the parts are proportional to the numbers S, $3S$, $5S$, $7S$... .



EXAMPLE 40 In the figure, the sides AB and BC are each divided into four equal parts.

Given that $A(AFIC) = 35\text{ cm}^2$,
find $A(\Delta ABC)$.



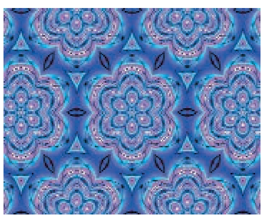
Solution By the previous rule,

$A(\Delta ABC) = S + 3S + 5S + 7S = 16S$.

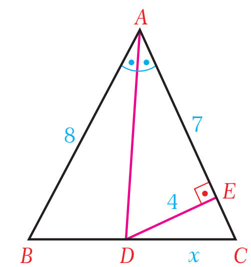
Also, $A(AFIC) = 7S = 35$, so $S = 5$.

As a result, $A(\Delta ABC) = 16S = 16 \cdot 5 = 80\text{ cm}^2$.

Check Yourself

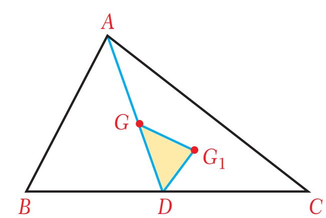


1. In the figure, AD is an angle bisector, $AB = 8$ cm, $AE = 7$ cm, $DE = 4$ cm and $A(\Delta ABC) = 36\text{ cm}^2$ are given. Find DC if $DE \perp AC$.



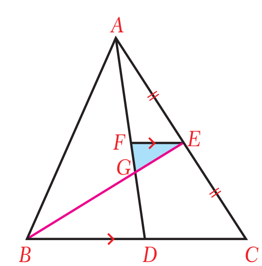
2. A triangle ABC has centroid G, and $A(\Delta AGC) = 8\text{ cm}^2$. Find $A(\Delta ABC)$.

3. In the figure, G is the centroid of ΔABC. Given that G_1 is the centroid of ΔADC and the area of ΔDGG_1 is 3, find $A(\Delta ABC)$.



4. The legs of a right triangle measure 6 cm and 8 cm. Find the inradius of this triangle.

5. In the figure, G is the centroid of ΔABC.
Given that E is the midpoint of AC,
$FE \parallel BC$ and $A(\Delta EFG) = 5$, find $A(\Delta ABC)$.

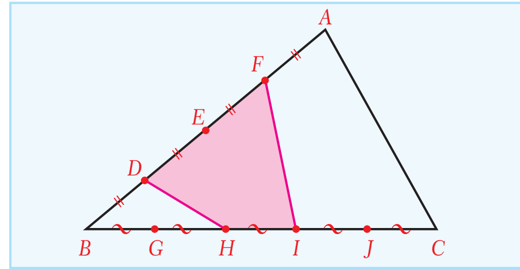


6. Two points D and E lie respectively on sides AB and AC of a triangle ABC. Given that $DE \parallel BC$, $AE = 2$ cm, $EC = 3$ cm and $A(BCED) = 42\text{ cm}^2$, find $A(\Delta ABC)$.

Answers

1. 5 cm 2. 24 cm^2 3. 54 4. 2 cm 5. 120 6. 50 cm^2

EXAMPLE 41 In the figure, AB is divided into four equal parts and BC is divided into five equal parts. If $A(\Delta ABC) = 180 \text{ cm}^2$, find $A(DHIF)$.



Solution First let us draw FH and AH, then we can use Property 4. In ΔABC the side BC is divided into five equal parts, so

$$A(\Delta ABH) = 2 \cdot \frac{180}{5} = 72 \text{ cm}^2.$$

In ΔABH, AB is divided into four equal parts, so $A(\Delta DFH) = 2 \cdot \frac{72}{4} = 36 \text{ cm}^2$.

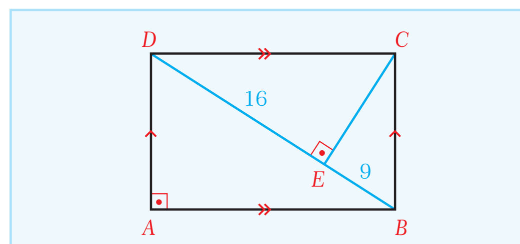
Now let us draw FC.

In ΔABC side AB is divided into four equal parts, so $A(\Delta BCF) = 3 \cdot \frac{180}{4} = 135 \text{ cm}^2$.

In ΔBCF side BC is divided into five equal parts, so $A(\Delta FHI) = \frac{135}{5} = 27 \text{ cm}^2$.

Finally, $A(DHIF) = A(\Delta DFH) + A(\Delta FHI) = 36 + 27 = 63 \text{ cm}^2$.

EXAMPLE 42 In the figure, $ABCD$ is a rectangle. Given that $CE \perp BD$, $BE = 9$ cm and $DE = 16$ cm, find $A(ABCD)$.



Solution By using the metric relations in a right triangle in ΔCDB we have

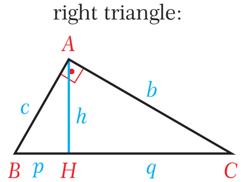
$CE^2 = DE \cdot BE$, i.e.

$CE^2 = 16 \cdot 9$ and $CE = 12$ cm.

Also, $A(\Delta BDC) = \frac{A(ABCD)}{2}$.

So $A(ABCD) = 2 \cdot A(\Delta BDC) = 2 \cdot \frac{12 \cdot (16+9)}{2} = 300 \text{ cm}^2$.

Metric relations in a right triangle:



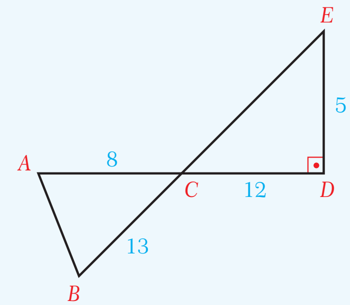
1. $h^2 = p \cdot q$
2. $c^2 = p \cdot a$
3. $b^2 = q \cdot a$

EXAMPLE 43 Prove that $A(\Delta ABC) = 2R^2 \sin A \cdot \sin B \cdot \sin C$, where R is the circumradius of ΔABC.

Solution By the law of sines, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, i.e. $a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$.

So $A(\Delta ABC) = \frac{a \cdot b \cdot c}{4R} = \frac{2R\sin A \cdot 2R\sin B \cdot 2R\sin C}{4R} = 2R^2 \sin A \cdot \sin B \cdot \sin C$, as required.

EXAMPLE 44 In the figure, ΔABC and ΔCDE are two triangles and $m\angle D = 90°$. Find $A(\Delta ABC)$.



Solution By the Pythagorean Theorem in ΔCDE,

$CE^2 = 5^2 + 12^2 = 169$, i.e. $CE = 13$.

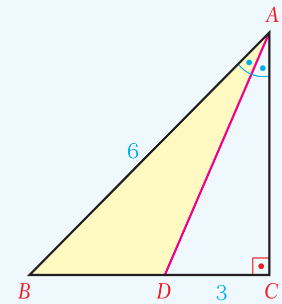
$\angle ACB$ and $\angle DCE$ are vertical angles so they are equal.

So $\sin(\angle ACB) = \sin(\angle DCE) = \dfrac{5}{13}$.

Finally, by the trigonometric formula for the area of a triangle we can write

$$A(\Delta ABC) = \frac{1}{2} \cdot AC \cdot BC \cdot \sin(\angle ACB) = \frac{1}{2} \cdot 8 \cdot 13 \cdot \frac{5}{13} = 20.$$

EXAMPLE 45 In the figure, ΔABC is a right triangle with $m\angle C = 90°$. Given that AD is an angle bisector, $AB = 6$ cm and $DC = 3$ cm, find $A(\Delta ABD)$.



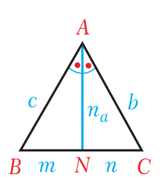
Solution Let $AC = h$ and $BD = x$.

By using the bisector theorem we have

$$\frac{AB}{BD} = \frac{AC}{DC}, \text{ i.e. } \frac{6}{x} = \frac{h}{3}, \; x \cdot h = 18.$$

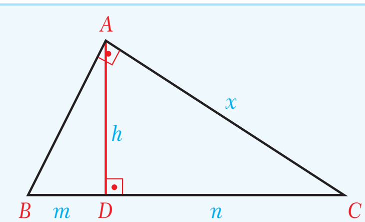
Finally, $A(\Delta ABD) = \dfrac{BD \cdot AC}{2} = \dfrac{x \cdot h}{2} = \dfrac{18}{2} = 9 \text{ cm}^2$.

Bisector theorem:



1. $\dfrac{c}{m} = \dfrac{b}{n}$
2. $n_a = \sqrt{(b \cdot c) - (m \cdot n)}$

EXAMPLE 46 In the figure, ΔABC is a right triangle. Given that $A(\Delta ABD) = 1 \text{ cm}^2$, $A(\Delta ADC) = 9 \text{ cm}^2$ and $AD \perp BC$, find the length $AC = x$.



Solution Let $AD = h$, $BD = m$ and $DC = n$.

By the metric relations in a right triangle we have $h^2 = m \cdot n$.

Also, $A(\Delta ABD) = 1 = \dfrac{m \cdot h}{2}$ and $A(\Delta ADC) = 9 = \dfrac{n \cdot h}{2}$,

which give us $m \cdot h = 2$ and $n \cdot h = 18$.

Multiplying these equations gives us $(m \cdot h) \cdot (n \cdot h) = 2 \cdot 18$, i.e. $m \cdot n \cdot h^2 = 36$.

But we know $h^2 = m \cdot n$, so $h^2 \cdot h^2 = 36$, $h = \sqrt{6}$ cm.

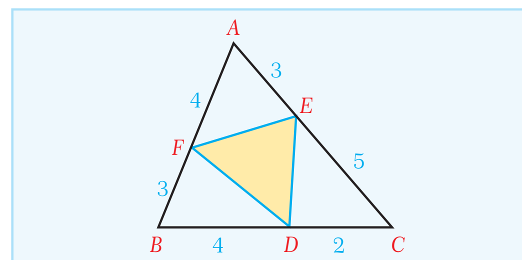
Using this in $n \cdot h = 18$ gives us $n = \dfrac{18}{\sqrt{6}} = 3\sqrt{6}$ cm.

Finally, the Pythagorean Theorem in ΔADC gives us $h^2 + n^2 = x^2$, i.e. $(\sqrt{6})^2 + (3\sqrt{6})^2 = x^2$,

which means $x^2 = 6 + 54 = 60$. So $x = \sqrt{60} = 2\sqrt{15}$ cm.

EXAMPLE 47 In the figure, $AF = 4$ cm, $FB = 3$ cm, $BD = 4$ cm, $DC = 2$ cm, $CE = 5$ cm and $EA = 3$ cm.

Find $\dfrac{A(\Delta DEF)}{A(\Delta ABC)}$.



Solution We can write

$$\frac{A(\Delta DEF)}{A(\Delta ABC)} = \frac{A(\Delta ABC) - A(\Delta AFE) - A(\Delta BDF) - A(\Delta CDE)}{A(\Delta ABC)}$$

$$= \frac{A(\Delta ABC)}{A(\Delta ABC)} - \frac{A(\Delta AFE)}{A(\Delta ABC)} - \frac{A(\Delta BDF)}{A(\Delta ABC)} - \frac{A(\Delta CDE)}{A(\Delta ABC)}$$

$$= 1 - \frac{A(\Delta AFE)}{A(\Delta ABC)} - \frac{A(\Delta BDF)}{A(\Delta ABC)} - \frac{A(\Delta CDE)}{A(\Delta ABC)}.$$

By the trigonometric formula for the area of a triangle,

$$\frac{A(\Delta AFE)}{A(\Delta ABC)} = \frac{\frac{1}{2} \cdot AF \cdot AE \cdot \sin A}{\frac{1}{2} \cdot AB \cdot AC \cdot \sin A} = \frac{AF \cdot AE}{AB \cdot AC} = \frac{4 \cdot 3}{7 \cdot 8} = \frac{3}{14}.$$

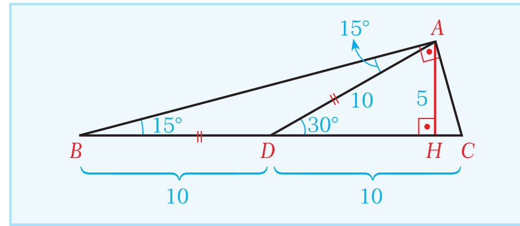
Similarly, $\dfrac{A(\Delta BDF)}{A(\Delta ABC)} = \dfrac{BD \cdot BF}{BC \cdot BA} = \dfrac{4 \cdot 3}{6 \cdot 7} = \dfrac{2}{7}$ and

$$\frac{A(\Delta CDE)}{A(\Delta ABC)} = \frac{CE \cdot CD}{CA \cdot CB} = \frac{5 \cdot 2}{8 \cdot 6} = \frac{5}{24}.$$

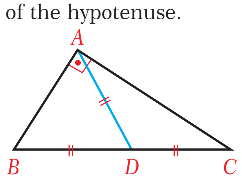
So $\dfrac{A(\Delta DEF)}{A(\Delta ABC)} = 1 - \dfrac{3}{14} - \dfrac{2}{7} - \dfrac{5}{24} = 1 - \dfrac{119}{168} = \dfrac{49}{168} = \dfrac{7}{24}.$

EXAMPLE 48 The hypotenuse of a right triangle measures 20 cm and its two acute angles are 15° and 75°. What is the area of this triangle?

Solution Look at the figure. ΔABC is the right triangle. Let us draw the median AD to the hypotenuse. By the property of a median to the hypotenuse, $AD = BD = CD = 10$ cm. If $AD = BD$ then $m(\angle ABD) = m(\angle BAD) = 15°$, which means $m(\angle ADC) = 30°$.



The median to the hypotenuse of a right triangle is half the length of the hypotenuse.

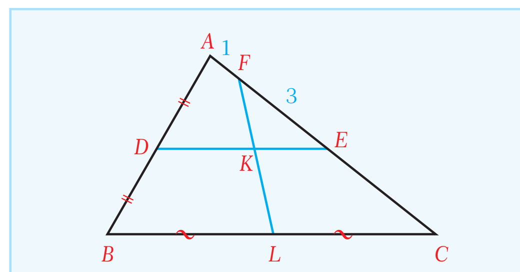


Now let us draw the altitude AH to the hypotenuse. In ΔADH, $m(\angle ADH) = 30°$ and $m(\angle AHD) = 90°$.

By the properties of a 30°-60°-90° triangle in ΔAHD, $AH = \dfrac{AD}{2} = \dfrac{10}{2} = 5$ cm.

So $A(\Delta ABC) = \dfrac{AH \cdot BC}{2} = \dfrac{5 \cdot 20}{2} = 50 \text{ cm}^2$.

EXAMPLE 49 In the figure, $DE \parallel BC$, D is the midpoint of AB and L is the midpoint of BC. Given that $AF = 1$ cm, $FE = 3$ cm and $A(EKLC) = 40 \text{ cm}^2$, find $A(\Delta ABC)$.



Solution If $DE \parallel BC$ and D is the midpoint of AB then E must be the midpoint of AC. So $EC = 4$ cm.

In the figure, $KE \parallel LC$ so ΔFKE and ΔFLC are similar triangles. By Property 9,

$\dfrac{A(\Delta FKE)}{A(\Delta FLC)} = \left(\dfrac{FE}{FC}\right)^2 = \left(\dfrac{3}{7}\right)^2 = \dfrac{9}{49}$. So $\dfrac{A(\Delta FKE)}{A(\Delta FKE)+40} = \dfrac{9}{49}$, i.e. $A(\Delta FKE) = 9 \text{ cm}^2$.

Now let us draw AL. In ΔALC, $FC = 7$ cm and $AF = 1$ cm. So $A(\Delta FLC) = 7S$ and $A(\Delta AFL) = S$.

We know $A(\Delta FLC) = A(\Delta FKE) + A(EKLC) = 9 + 40 = 49 = 7S$, so $A(\Delta AFL) = S = 7 \text{ cm}^2$.

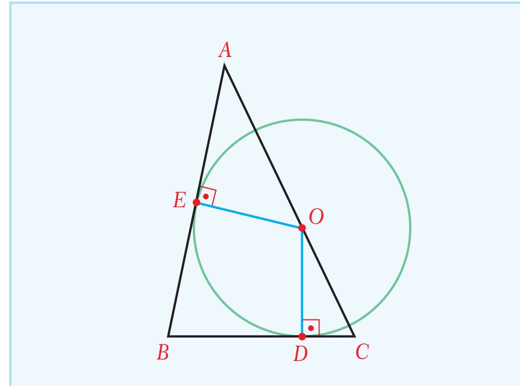
So $A(ALC) = 8S = 56 \text{ cm}^2$. L is the midpoint of BC, so $A(\Delta ALC) = A(\Delta ABL) = 56 \text{ cm}^2$.

So $A(\Delta ABC) = A(\Delta ABL) + A(\Delta ALC) = 56 + 56 = 112 \text{ cm}^2$.

EXAMPLE 50 ΔABC is a triangle with sides $a = 6$ cm, $b = 7$ cm and $c = 5$ cm. A circle is drawn which is centered on AC and tangent to the sides AB and BC. Find the radius of this circle.

Solution The figure illustrates the problem. Let us draw the radii from point O to the points of tangency on sides BC and AB. If D and E are the points of tangency then we can say that $OD \perp BC$ and $OE \perp AB$, since a line passing through the center of a circle is perpendicular to any tangent line at the point of tangency.

If we draw OB then $A(\Delta ABC) = A(\Delta ABO) + A(\Delta BOC) = \dfrac{AB \cdot OE}{2} + \dfrac{BC \cdot OD}{2} = \dfrac{5r}{2} + \dfrac{6r}{2} = \dfrac{11r}{2}$.



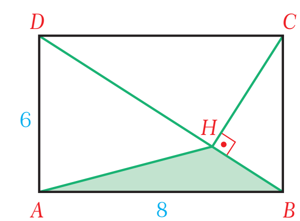
We can calculate $A(\Delta ABC)$ by using Heron's Formula with $u = \dfrac{6+7+5}{2} = 9$:

$$A(\Delta ABC) = \sqrt{9 \cdot (9-6) \cdot (9-7) \cdot (9-5)}$$
$$= 6\sqrt{6} \text{ cm}^2.$$

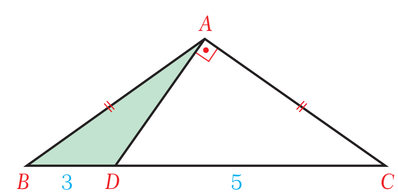
So $A(\Delta ABC) = \dfrac{11r}{2} = 6\sqrt{6}$ and $r = \dfrac{12\sqrt{6}}{11}$ cm.

Check Yourself

1. In the figure, $ABCD$ is a rectangle, $AB = 8$ cm, $AD = 6$ cm and $CH \perp BD$. Find $A(\Delta ABH)$.

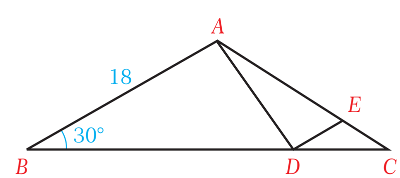


2. In the figure, ABC is an isosceles triangle with $AB = AC$. Given that $m(\angle DAC) = 90°$, $BD = 3$ cm and $DC = 5$ cm, find $A(\Delta ABD)$.



3. $ABCD$ is a quadrilateral with $m(\angle A) = 90°$. Given that $AD = 12$, $AB = 16$, $BC = 20$ and $CD = 24$, find $A(ABCD)$.

4. In the figure, $AE = 5 \cdot EC$, $BC = 5 \cdot DC$, $m(\angle B) = 30°$ and $AB = 18$ cm. If $A(\Delta ADE) = 15$ cm^2, find BD.

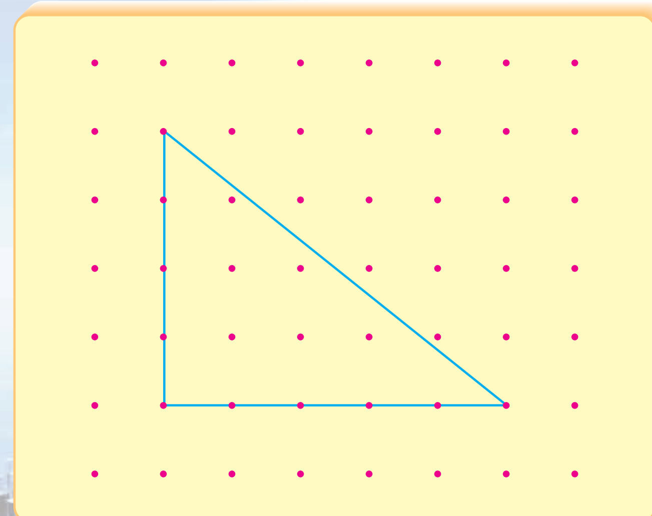


Answers

1. $\dfrac{216}{25}$ cm^2 2. 3 cm^2 3. 288 4. 16 cm

PICK'S THEOREM

Throughout history, people have developed different ways of finding the area of a polygonal region. One of these people was Georg Alexander Pick. He was born in Vienna in 1859 and died in 1943 in a concentration camp. His famous theorem helps us to find the area of a polygonal region whose vertices are points in a regular square grid, such as the region shown opposite. The distance between points in the grid must be one unit.



Pick's Theorem tells us that if I is the number of grid points inside the polygon and B is the number of points on its perimeter, the area is

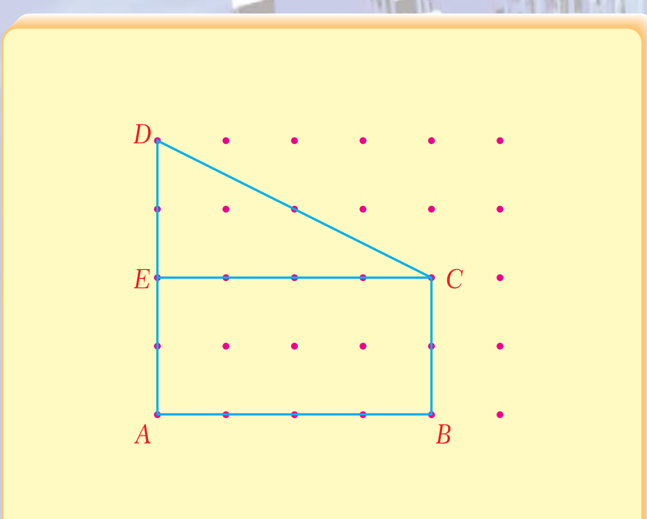
$$\boxed{Area = I + \frac{B}{2} - 1}\ .$$

In other words, the area is one less than the sum of the interior points and half of the points on the boundary.

Let us look at a simple example. Look at the triangle above right. Its legs are four and five units long, so using regular geometry we can say that its area is $A = \frac{4 \cdot 5}{2} = 10$ square units.

Alternatively, using Pick's Theorem with $I = 6$ and $B = 10$ gives us

$A = I + \frac{B}{2} - 1 = 6 + \frac{10}{2} - 1 = 10.$



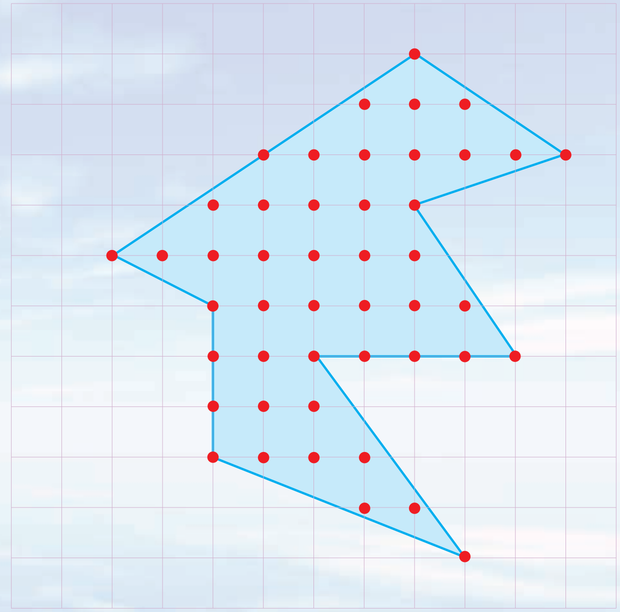
Now look at the figure on the left. Using our knowledge of geometry, the combined area of the triangle and rectangle is

$$A = A_{\text{triangle}} + A_{\text{rectangle}}$$

$$= \frac{4 \cdot 2}{2} + (4 \cdot 2) = 4 + 8$$

$$= 12 \text{ square units.}$$

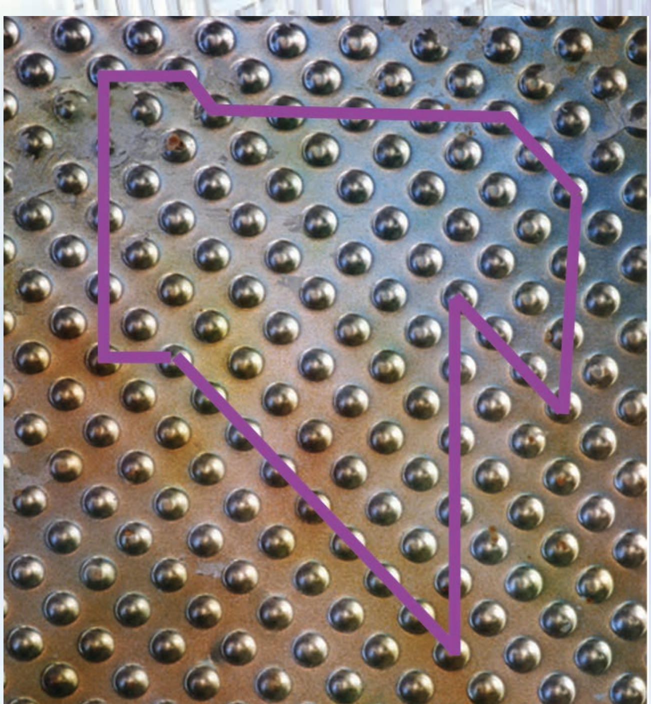
You can check this with Pick's Theorem using $I = 5$ and $B = 10$.

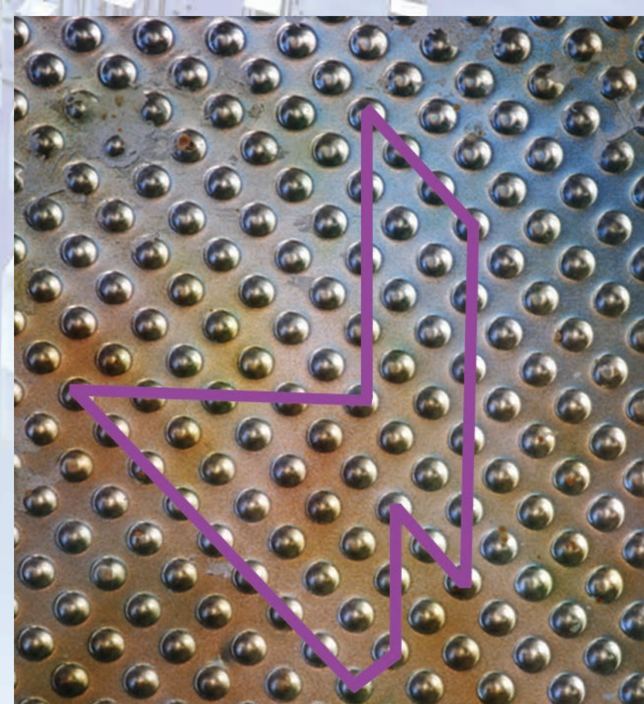


Pick's Theorem is especially useful for finding the area of complicated shapes. Look at the shape on the right. We could divide it into rectangles and triangles and calculate their area, but this would take a long time. So we can use Pick's Theorem with $I = 31$ and $B = 15$:

$A = I + \frac{B}{2} - 1 = 31 + \frac{15}{2} - 1 = 37.5$ square units.

Of course, we cannot find the areas of all shapes using Pick's Theorem. Remember that the vertices of a shape must all be points in a square grid before we can use the theorem. So for example, we cannot use it to calculate the area of an equilateral triangle, because the three vertices of an equilateral triangle will never all lie at points on a square grid. We also know that the area of an equilateral triangle involves square roots, and Pick's Theorem does not use square roots. Therefore the theorem is only useful in some cases.





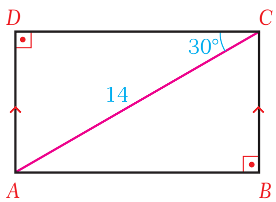
EXERCISES 3.2

A. The Concept of Area

1. A rectangle has perimeter 80 and one side is four times as long as the other side. Find the area of this rectangle.

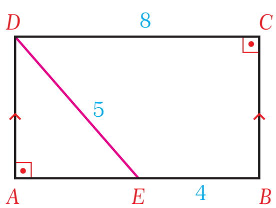
2. The diagonal of a rectangle measures 20 units. If one of the sides is 12 units long, find the area of this rectangle.

3. In the figure, $ABCD$ is a rectangle. Given that $AC = 14$ and $m(\angle ACD) = 30°$, find the area of this rectangle.



4. The ratio of the sides of a rectangle is 3 : 5 and its area is 240 square units. Find the perimeter of the rectangle.

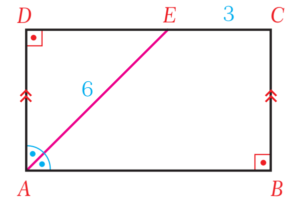
5. In the figure, $ABCD$ is a rectangle. Given that $CD = 8$, $DE = 5$ and $EB = 4$, find the area of the rectangle.



6. A rectangle has area 72 unit^2 and perimeter 34 units. Find the lengths of the sides of this rectangle.

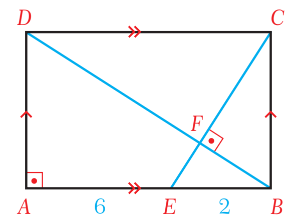
7. A rectangle has area 84 unit^2. Given that one of the sides is five units longer than the other side, find the perimeter of this rectangle.

8. In the figure, $ABCD$ is a rectangle. Given that AE is the bisector of $\angle A$, $AE = 6$ and $EC = 3$, find the area of $ABCD$.



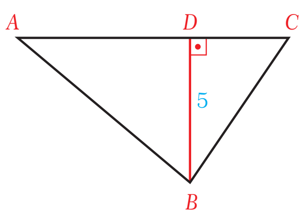
9. One side of a rectangle is three times as long as the other side and its diagonal measures 6 units. Find the area of this rectangle.

10. ✪ In the figure, $ABCD$ is a rectangle. Given that $CE \perp BD$, $AE = 6$ and $EB = 2$, find $A(ABCD)$.



B. Area of a Triangle

11. In the figure, $AC = 12$ and $BD = 5$. Given that $BD \perp AC$, find $A(\Delta ABC)$.



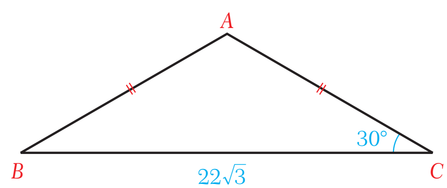
12. A triangle ΔABC has sides $a = 6$ and $b = 8$. If $h_a = 10$, find h_b.

13. The base of an isosceles triangle is 10 units long and its other sides are each 13 units long. Find the area of this triangle.

14. In ΔABC, $BC = 8$, $AC = 6$ and $m(\angle C) = 45°$. Find $A(\Delta ABC)$.

15. The sides of a triangle ABC are $a = 13$, $b = 14$ and $c = 15$.
Given that $A(\Delta ABC) = 84$, find the lengths of the three altitudes of the triangle.

16.



In the figure, ΔABC is an isosceles triangle with $AB = AC$. Given that $m(\angle C) = 30°$ and the length of the base is $BC = 22\sqrt{3}$, find $A(\Delta ABC)$.

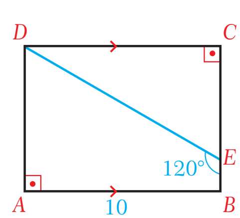
17. In a right triangle ΔABC the hypotenuse BC is 17 units long and one of the legs is 15 units long. Find $A(\Delta ABC)$.

18. The area of an isosceles right triangle is 16 square units. Find the hypotenuse of this triangle.

19. The perimeter of a right triangle is 56 units and its hypotenuse is 25 units. Find the area of this triangle.

20. A right triangle has hypotenuse $AC = 10$. If one of the acute angles of this triangle is 30°, find the area of the triangle.

21. In the figure, $ABCD$ is a rectangle.
Given that $m(\angle BED) = 120°$ and $AB = 10$ cm, find the area of ΔCDE.



22. One of the acute angles in a right triangle measures 22.5° and the length of the hypotenuse is 12 units. Find the area of this triangle.

23. Find the area of the equilateral triangle with the given side length.

a. 12 **b.** $4\sqrt{3}$ **c.** $3\sqrt{2}$

24. Find the area of the equilateral triangle with the given height.

a. $3\sqrt{3}$ **b.** 4 **c.** $6\sqrt{5}$

25. The altitude to the hypotenuse of a right triangle divides the hypotenuse into two parts of lengths 4 and 9 units. Find the area of this triangle.

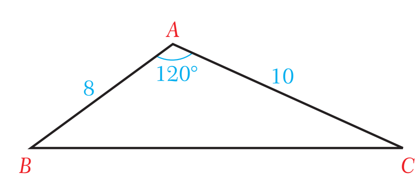
26. Find the area of the triangle with the given side lengths.

a. 11, 12 and 15

b. 7, 9 and 10

27. The sides of a triangle are $a = 12$, $b = 13$ and $c = 15$. Find h_b.

28.

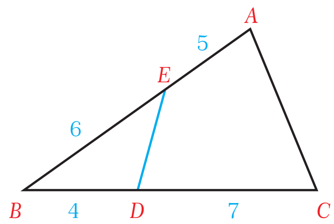


In the figure, $m(\angle A) = 120°$. Given that $AB = 8$ and $AC = 10$, find $A(\Delta ABC)$.

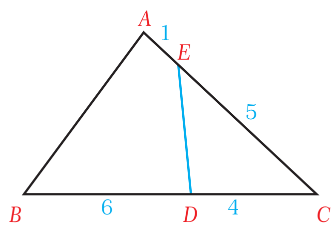
29. Two adjacent sides of a triangle are 6 and 8 units long. Find the area of this triangle if the angle between these two sides is

a. 30°. **b.** 45°. **c.** 90°. **d.** 120°.

30. In the figure,
$BD = 4$,
$DC = 7$,
$BE = 6$ and
$EA = 5$.
If $A(\Delta BDE) = 9$, what is $A(\Delta ABC)$?



31. In the figure,
$AE = 1$,
$EC = 5$,
$BD = 6$ and
$DC = 4$.
Find $\dfrac{A(\Delta ABC)}{A(ABDE)}$.



32. The sides of a triangle are $a = 10$, $b = 17$ and $c = 11$. Find the value of $\sin C$.

33. A triangle has perimeter 24 units. Given that the area of this triangle is 60 square units, find its inradius.

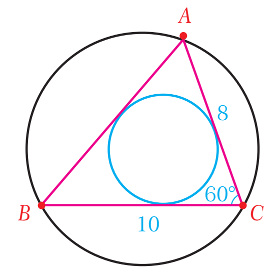
34. The legs of a right triangle are 9 and 12 units long. Find the inradius and circumradius of this triangle.

35. The sides of an isosceles triangle measure 16, 10 and 10 units. Find the sum of its inradius and circumradius.

36. One side of an equilateral triangle is 8 units long. Find the inradius r and circumradius R of this triangle.

37. The circumradius of a triangle is 8 units. Given that $a = 10$, find the value of $\sin A$.

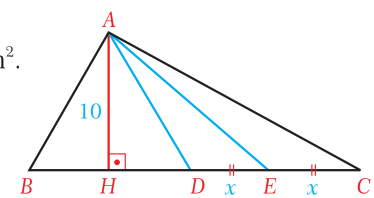
38. ✪ In the figure,
$AC = 8$,
$BC = 10$ and
$m(\angle C) = 60°$.
Find the inradius r and circumradius R of ΔABC.



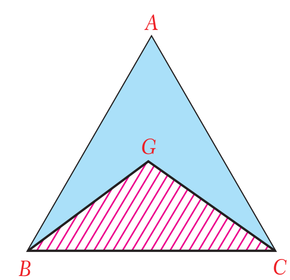
C. Properties of the Area of a Triangle

39. Find the length of the altitude to the hypotenuse in a right triangle with legs 8 and 15 units long.

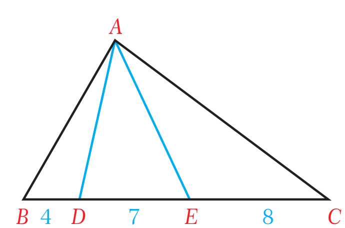
40. In the figure,
$A(\Delta ABC) = 85\ \text{cm}^2$.
Given that
$BD = 9$,
$AH = 10$ and
$DE = EC = x$,
find x.



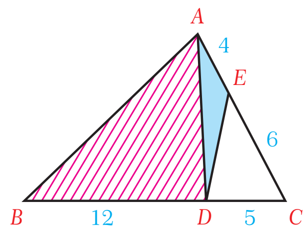
41. In the figure, G is the centroid of ΔABC. Given that $A(ABGC) = 20$, find $A(\Delta BCG)$.



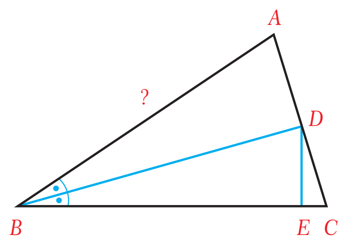
42. In the figure,
$BD = 4$,
$DE = 7$ and
$EC = 8$.
Given that
$A(\Delta ABC) = 95$,
find $A(\Delta ABD) + A(\Delta AEC)$.



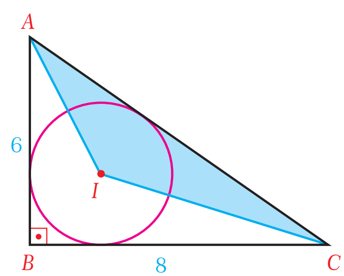
43. In the given figure,
$BD = 12$,
$DC = 5$,
$CE = 6$ and
$EA = 4$.
The area of ΔAED is 12.
What is the area of ΔABD?



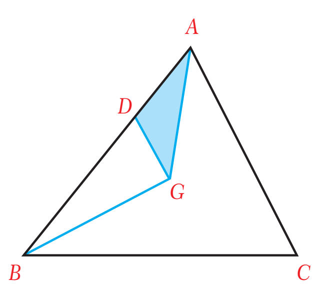
44. In the figure, BD bisects $\angle B$,
$DE \perp BC$,
$DE = 4$ and
$A(\Delta ABD) = 24$.
Find the length of AB.



45. In the figure, I is the center of the incircle of the right triangle ΔABC.
The legs of ΔABC are 6 and 8 units long. Find $A(\Delta AIC)$.

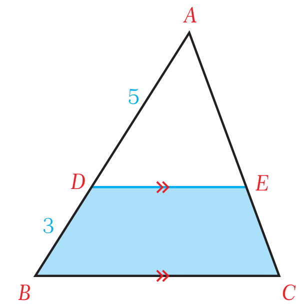


46. In the figure, G is the centroid of ΔABC.
$AB = 4 \cdot AD$ and
$A(\Delta ABC) = 84$
are given.
Find $A(\Delta ADG)$.

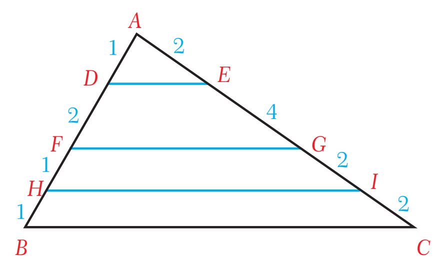


47. The lengths of the altitudes of a triangle are 3, 4 and 6 units. Find the inradius of this triangle.

48. In the figure,
$DE \parallel BC$.
Given that
$AD = 5$,
$DB = 3$ and
$A(DBCE) = 12$,
find $A(\Delta ABC)$.

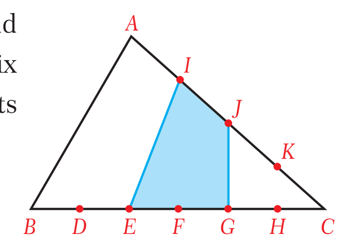


49.



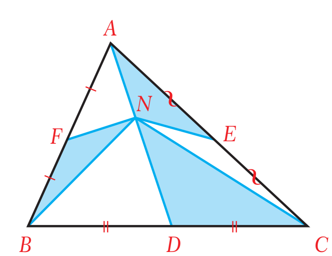
In the figure, $AD = FH = HB = 1$ cm, $DF = AE = GI = IC = 2$ and $EG = 4$. Given that $A(DFGE) = 12$, find $A(\Delta ABC)$.

50. In the figure, BC and AC are divided into six and four equal parts respectively.
$A(\Delta ABC) = 120$ is given. Find $A(EGJI)$.

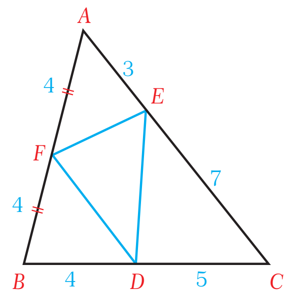


51. The sides of a triangle measure 12, 14 and 16 units. Find the inradius r and circumradius R of this triangle.

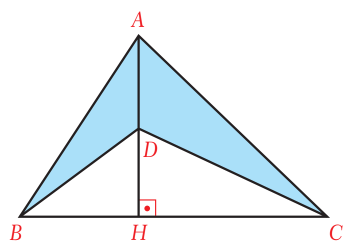
52. In the figure, AD is a median and E and F are the midpoints of AC and AB respectively.
If the sum of the shaded areas is 12, find $A(\Delta ABC)$.



53. In the figure,
$AF = FB = BD = 4$,
$DC = 5$,
$EC = 7$ and
$AE = 3$.
Find $\dfrac{A(\Delta ABC)}{A(\Delta DEF)}$.



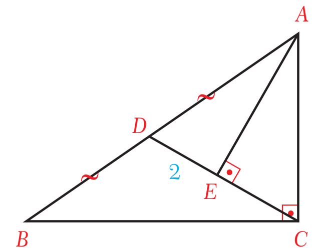
54. In the figure, point D lies on the altitude AH.
Given that
$BC = 10$ and
$AD = 6$,
find $A(ABDC)$.



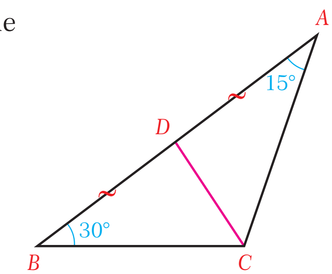
55. An equilateral triangle ABC is given. Point P lies on the base BC such that $m(\angle APB) = 75°$. If one side of the triangle is 12 units long, find $A(\Delta APC)$.

56. P is a point in the interior of an equilateral triangle such that the distances from P to the vertices are 5, 12 and 13 units. Find the area of this triangle.

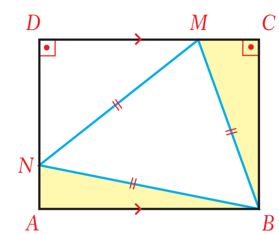
57. In the figure, ΔABC is a right triangle with $m(\angle C) = 90°$.
Given that CD is the median of side AB,
$CD \perp AE$,
$ED = 2$ and
$AE = \dfrac{BC}{2}$, find $A(\Delta ABC)$.



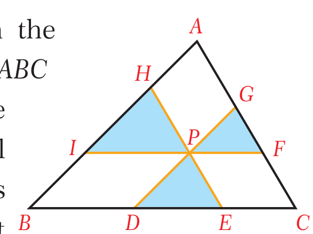
58. In the figure, CD is the median of side AB.
Given that
$AB = 10$,
$m(\angle ABC) = 30°$ and
$m(\angle BAC) = 15°$,
find $A(\Delta ABC)$.



59. In the figure, $ABCD$ is a rectangle and points M and N lie on the sides CD and AD, respectively. Given that ΔBMN is an equilateral triangle and $A(\Delta DMN) = 12$, find $A(\Delta ABN) + A(\Delta BCM)$.



60. A point P is taken in the interior of a triangle ABC and through it three lines are drawn parallel to the sides of ΔABC, as in the figure. Prove that



$$\sqrt{A(\Delta ABC)} = \sqrt{A(\Delta HIP)} + \sqrt{A(\Delta DEP)} + \sqrt{A(\Delta FGP)}.$$

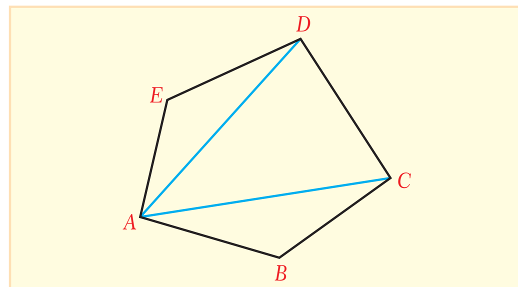
In the previous section we studied the areas of rectangles and triangles. In this section we will use what we have learned to begin our study of quadrilaterals. First we will study the area of a general quadrilateral, and then we will look at the areas of special quadrilaterals such as parallelograms and rhombi.

D. AREA OF A QUADRILATERAL

Rule

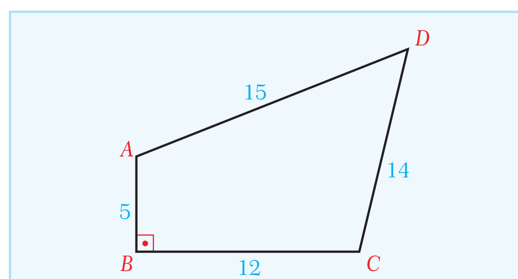
By drawing the diagonals of a polygon from one of its vertices, we can divide the area of the polygon into small triangles and then use the areas of the triangles to calculate the area of the given region.

For example, in the figure,



$$A(ABCDE) = A(\Delta ABC) + A(\Delta ACD) + A(\Delta ADE).$$

EXAMPLE 51 In the figure, $AB = 5$ cm, $BC = 12$ cm, $CD = 14$ cm and $AD = 15$ cm. Find the area of the quadrilateral $ABCD$ if $m(\angle ABC) = 90°$.



Solution Let us join A and C to get two triangles, ΔABC and ΔACD.

ΔABC is a right triangle, so $A(\Delta ABC) = \dfrac{AB \cdot BC}{2} = \dfrac{5 \cdot 12}{2} = 30 \text{ cm}^2$.

Also, the Pythagorean Theorem in ΔABC gives

$AC^2 = 5^2 + 12^2 = 25 + 144 = 169$, i.e. $AC = 13$ cm.

Now let us use Heron's Formula in ΔACD with $u = \dfrac{13+14+15}{2} = 21$:

$$\Delta ACB = \sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84 \text{ cm}^2.$$

So $A(ABCD) = A(\Delta ABC) + A(\Delta ACD)$

$= 30 + 84$

$= 114 \text{ cm}^2$.

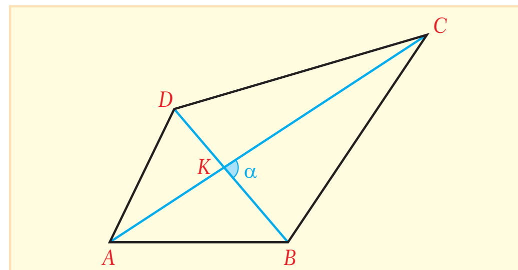
Theorem

area of a convex polygon

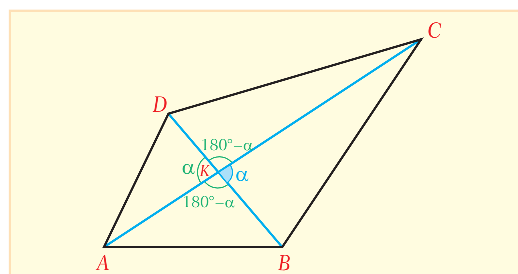
If all the interior angles of a polygon are smaller than 180° then the polygon is a convex polygon.

The area of a convex polygon is equal to half the product of the lengths of diagonals and the sine of the angle between the diagonals: in the figure,

$$\boxed{A(ABCD) = \frac{AC \cdot BD \cdot \sin\alpha}{2}}\ .$$



Proof Let K be the intersection point of the diagonals.



From the figure,

$A(ABCD) = A(\Delta ABK) + A(\Delta BCK) + A(\Delta CDK) + A(\Delta ADK)$. So by the trigonometric formula,

$$A(ABCD) = \frac{AK \cdot BK \cdot \sin(180° - \alpha)}{2} + \frac{BK \cdot CK \cdot \sin\alpha}{2} + \frac{CK \cdot DK \cdot \sin(180° - \alpha)}{2} + \frac{AK \cdot DK \cdot \sin\alpha}{2}$$

We know that $\sin\alpha = \sin(180° - \alpha)$, so we can reduce the above expression to

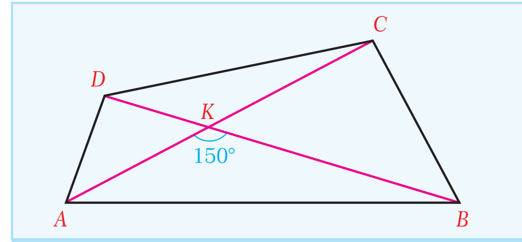
$$A(ABCD) = \frac{\sin\alpha}{2} \cdot [(AK \cdot BK) + (BK \cdot CK) + (CK \cdot DK) + (AK \cdot DK)]$$

$$= \frac{\sin\alpha}{2} \cdot [[(AK + CK) \cdot BK] + [(CK + AK) \cdot DK)]]$$

$$= \frac{\sin\alpha}{2} \cdot ((AK + CK) \cdot (BK + DK))$$

$$= \frac{AC \cdot BD \cdot \sin\alpha}{2}. \qquad (AK + CK = AC \text{ and } BK + DK = BD)$$

EXAMPLE 52 In the figure, $AC = 12$ cm, $BD = 15$ cm and $m(\angle AKB) = 150°$. Find the area of the quadrilateral $ABCD$.



Solution $A(ABCD) = \dfrac{AC \cdot BD \cdot \sin\alpha}{2}$, so

$$A(ABCD) = \frac{12 \cdot 15 \cdot \sin 150°}{2}$$

$$= 6 \cdot 15 \cdot \frac{1}{2}$$

$$= 45 \text{ cm}^2.$$

Note

If the diagonals of a quadrilateral are perpendicular to each other then the formula for its area becomes

$$\boxed{A(ABCD) = \frac{AC \cdot BD}{2}}$$

since $\sin 90° = 1$.

EXAMPLE 53 The diagonals of a quadrilateral $ABCD$ are perpendicular to each other with $AC = 3 \cdot BD$. If the area of $ABCD$ is 48 cm^2, find the lengths of the diagonals.

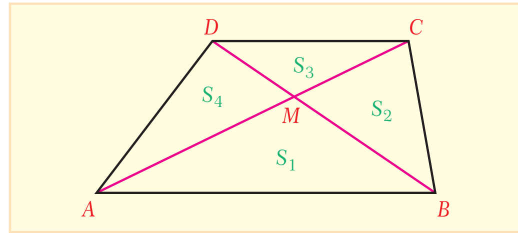
Solution Let $BD = x$, so $AC = 3x$.

Using $A(ABCD) = \dfrac{AC \cdot BD}{2}$ gives $48 = \dfrac{x \cdot 3x}{2}$.

So $x^2 = 32$, $x = 4\sqrt{2}$. So the diagonals are $BD = x = 4\sqrt{2}$ cm and $AC = 3x = 12\sqrt{2}$ cm.

Theorem

The diagonals of a quadrilateral $ABCD$ divide the area of the quadrilateral into four parts as shown in the figure. If



$A(\Delta ABM) = S_1$,

$A(\Delta BCM) = S_2$,

$A(\Delta CDM) = S_3$ and

$A(\Delta ADM) = S_4$, then

$$\boxed{S_1 \cdot S_3 = S_2 \cdot S_4}.$$

Proof We know from Property 4 that if two triangles are the same height then the ratio of their areas is the same as the ratio of their bases.

So we can write

$$\frac{A(\Delta ABM)}{A(\Delta ADM)} = \frac{S_1}{S_4} = \frac{BM}{DM} \text{ and}$$

$$\frac{A(\Delta BCM)}{A(\Delta DCM)} = \frac{S_2}{S_3} = \frac{BM}{DM}.$$

So $\dfrac{BM}{DM} = \dfrac{S_1}{S_4} = \dfrac{S_2}{S_3}$.

Cross multiplying gives us

$S_1 \cdot S_3 = S_2 \cdot S_4$, as required.

EXAMPLE 54 AC and BD are the diagonals of a quadrilateral $ABCD$, and M is their point of intersection. If $A(\Delta ABM) = 18\text{ cm}^2$, $A(\Delta CDM) = 12\text{ cm}^2$ and $A(\Delta BCM) = A(\Delta ADM)$, find $A(ABCD)$.

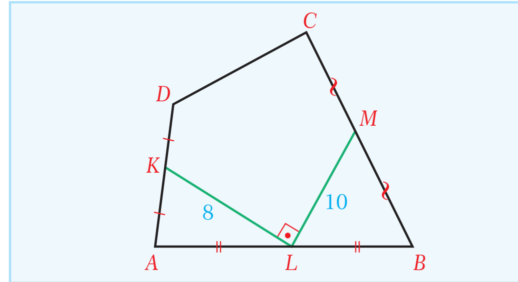
Solution Let $S_1 = 18$, $S_3 = 12$, $S_2 = S_4 = x$.

By the previous theorem, $S_1 \cdot S_3 = S_2 \cdot S_4$, so

$18 \cdot 12 = x \cdot x$, $x^2 = 216\text{ cm}^2$, $x = 6\sqrt{6}\text{ cm}^2$. So

$A(ABCD) = S_1 + S_2 + S_3 + S_4 = 18 + 6\sqrt{6} + 12 + 6\sqrt{6} = (30 + 12\sqrt{6})\text{ cm}^2$.

EXAMPLE 55 In the figure, $ABCD$ is a quadrilateral and K, L and M are the midpoints of their respective sides. Given that $KL \perp LM$, $KL = 8$ cm and $LM = 10$ cm, find the area of quadrilateral.



Solution Let us draw the diagonals AC and BD. In ΔABD points K and L are the midpoints so KL is a midsegment. By the properties of a midsegment,

$KL \parallel BD$ and $BD = 2 \cdot KL = 2 \cdot 8 = 16$ cm.

Also, if $KL \parallel BD$ then

$m(\angle LNR) = m(\angle KLN) = 90°$.

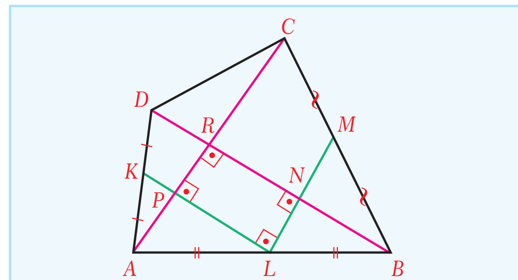
In ΔABC points L and M are midpoints so LM is a midsegment, which gives us $AC \parallel LM$ and $AC = 2 \cdot LM = 2 \cdot 10 = 20$ cm.

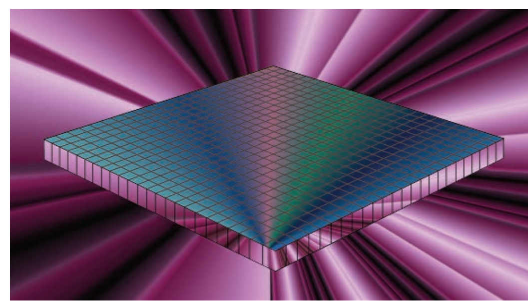
If $AC \parallel LM$ then $m(\angle ARB) = m(\angle LNR) = 90°$.

As a result, $AC \perp BD$ and so

$$A(ABCD) = \frac{AC \cdot BD}{2} = \frac{16 \cdot 20}{2} = 160 \text{ cm}^2.$$

A midsegment of a triangle is a line segment that connects the midpoints of two sides of the triangle.





Check Yourself

1. In the quadrilateral $ABCD$, $m(\angle DAB) = m(\angle BCD) = 90°$. If $AD = 12$ cm, $AB = 16$ cm and $BC = 10$ cm, find $A(ABCD)$.
2. In a quadrilateral $ABCD$, point E is the intersection point of the diagonals and AC is perpendicular to BD. Given that $DE = AE = EC = 6$ cm and $AB = 10$ cm, find the area of $ABCD$.
3. AC and BD are the diagonals of a quadrilateral such that $AC = 4 \cdot BD$ and the angle between them is 30°. If the area of this quadrilateral is 160 unit^2, find the lengths of AC and BD.
4. The diagonals AC and BD of the convex quadrilateral $ABCD$ intersect at point E. $A(\Delta ABE) = 12$ cm^2, $A(\Delta CDE) = 16$ cm^2 and $A(\Delta BCE) = 3 \cdot A(\Delta ADE)$ are given. Find $A(ABCD)$.

Answers

1. $(96 + 50\sqrt{3})$ cm^2 2. 84 cm^2 3. $AC = 16\sqrt{10}$, $BD = 4\sqrt{10}$ 4. 60 cm^2

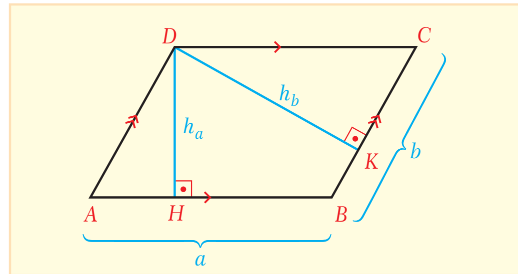
E. AREA OF A PARALLELOGRAM

Theorem

area of a parallelogram

The area of a parallelogram is the product of the length of any base and the length of the corresponding altitude: in the figure,

$$\boxed{A(ABCD) = a \cdot h_a = b \cdot h_b}\ .$$



A parallelogram is a quadrilateral whose opposite sides are congruent and parallel to each other.

Proof If we draw the diagonal BD we get two congruent triangles, $\Delta ABD \cong \Delta CDB$. So

$$\begin{aligned} A(ABCD) &= A(\Delta ABD) + A(\Delta CDB) \\ &= 2 \cdot A(\Delta ABD) \\ &= 2 \cdot \frac{1}{2} \cdot a \cdot h_a \\ &= a \cdot h_a. \end{aligned}$$

We can use similar reasoning to show $A = b \cdot h_b$.

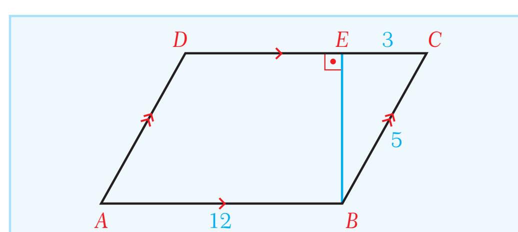
EXAMPLE 56 Two sides of a parallelogram measure 6 cm and 8 cm. The height from the shorter side is 12 cm. Find the height from the longer side.

Solution Since this is a parallelogram,

$A = a \cdot h_a = b \cdot h_b$, i.e. $6 \cdot 12 = 8 \cdot h_b$.

So $h_b = \dfrac{6 \cdot 12}{8} = 9$ cm.

EXAMPLE 57 In the figure, $ABCD$ is a parallelogram and BE is perpendicular to DC.
Given that $AB = 12$ cm, $EC = 3$ cm and $BC = 5$ cm, find $A(ABCD)$.



Solution $A(ABCD) = AB \cdot h_a = AB \cdot BE$.

We need to find BE.

By the Pythagorean Theorem in ΔBEC we have $BE^2 + 3^2 = 5^2$, $BE = 4$ cm.

So $AB = 12$ cm and $BE = h_a = 4$ cm. So $A(ABCD) = 12 \cdot 4 = 48$ cm^2.

EXAMPLE 58 $ABCD$ is a parallelogram with sides $AD = 8$ cm and $AB = 12$ cm. Given that $m(\angle ABC) = 150°$, find the area of this parallelogram.

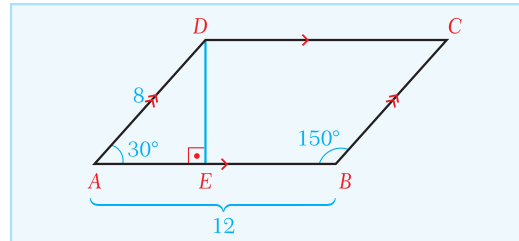
Solution If $m(\angle ABC) = 150°$ then $m(\angle A) = 30°$.

Let us draw the altitude from the vertex D to AB and let E be the foot of this altitude.

In ΔAED, $AD = 8$ cm so

$DE = 4$ cm (since this is a 30°-60°-90° triangle).

So $A(ABCD) = 12 \cdot 4 = 48$ cm^2.

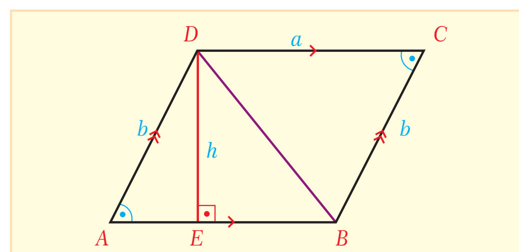


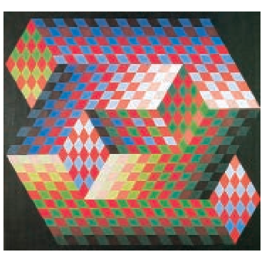
Theorem

If $ABCD$ is a parallelogram with sides a and b separated by an angle A then

$A(ABCD) = a \cdot b \cdot \sin A$.

Proof Let us write $AB = CD = a$ and $BC = AD = b$.





Then we draw the altitude DE from vertex D to side AB with length h.

From the figure, $\sin A = \dfrac{h}{b}$, i.e.

$h = b \cdot \sin A$.

So $A(ABCD) = a \cdot h = a \cdot b \cdot \sin A$.

EXAMPLE 59 The sides of a parallelogram measure 6 cm and 8 cm. Find its area if its interior angles measure 60° and 120°.

Solution We can use the theorem above with either sin 60° or sin 120°, since we know

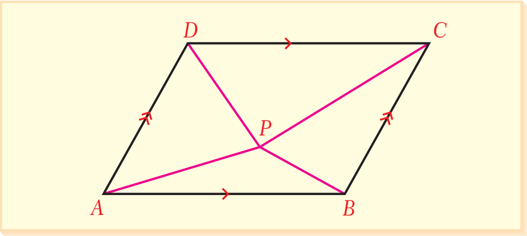
$\sin 60° = \sin 120° = \dfrac{\sqrt{3}}{2}$.

So $A(ABCD) = a \cdot b \cdot \sin A = 6 \cdot 8 \cdot \sin 60° = 6 \cdot 8 \cdot \dfrac{\sqrt{3}}{2} = 24\sqrt{3}$ cm^2.

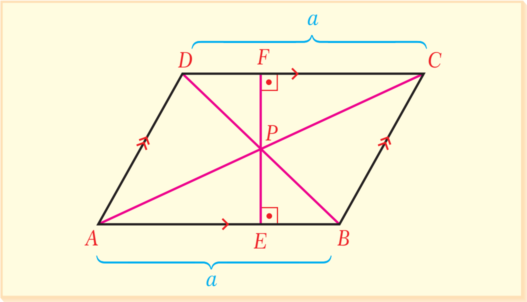
Theorem

If P is any point inside a parallelogram $ABCD$ then

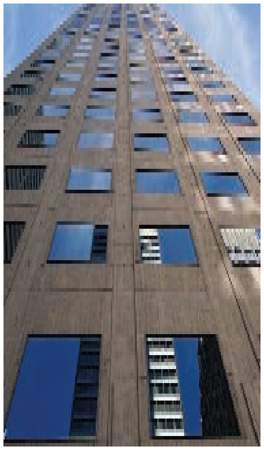
$$A(\Delta PAB) + A(\Delta PCD) = A(\Delta PBC) + A(\Delta PDA) = \frac{A(ABCD)}{2}.$$



Proof Let us draw the altitudes PE and PF from point P to sides AB and CD respectively.



Then $PE + PF = h_a$.



Also, $A(\Delta PAB) + A(\Delta PCD) = \frac{a \cdot PE}{2} + \frac{a \cdot PF}{2}$

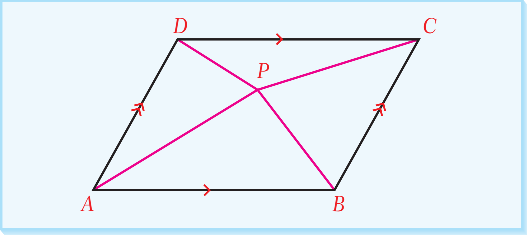
$$= \frac{a \cdot (PE + PF)}{2}$$

$$= \frac{a \cdot h_a}{2}$$

$$= \frac{A(ABCD)}{2}.$$

In the same way we can prove that $A(\Delta PBC) + A(\Delta PDA) = \frac{A(ABCD)}{2}$.

EXAMPLE 60 In the figure, P is a point inside the parallelogram $ABCD$. Given that $A(\Delta PAB) = 13 \text{ cm}^2$, $A(\Delta PCD) = 12 \text{ cm}^2$ and $\frac{A(\Delta PBC)}{A(\Delta PDA)} = \frac{2}{3}$, find $A(\Delta PBC)$, $A(\Delta PDA)$ and $A(ABCD)$.



Solution We are given $\frac{A(\Delta PBC)}{A(\Delta PDA)} = \frac{2}{3}$, so we can write $A(\Delta PBC) = 2S$ and $A(\Delta PDA) = 3S$.

By the previous theorem,

$$A(\Delta PAB) + A(\Delta PCD) = A(\Delta PBC) + A(\Delta PDA) = \frac{A(ABCD)}{2}$$

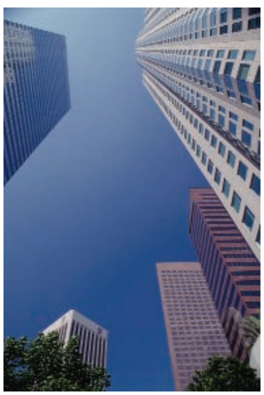
$$13 + 12 = 2S + 3S, \text{ i.e. } 5S = 25 \text{ and } S = 5.$$

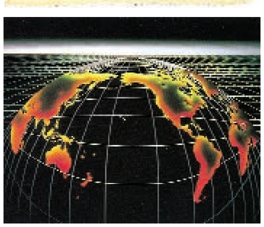
So $A(\Delta PBC) = 2S = 2 \cdot 5 = 10\ \text{cm}^2$,

$A(\Delta PDA) = 3S = 3 \cdot 5 = 15\ \text{cm}^2$ and

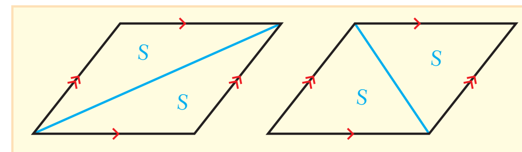
$A(ABCD) = 2 \cdot (A(\Delta PAB) + A(\Delta PCD)) = 2 \cdot (13 + 12) = 2 \cdot 25 = 50\ \text{cm}^2$.

Properties 10

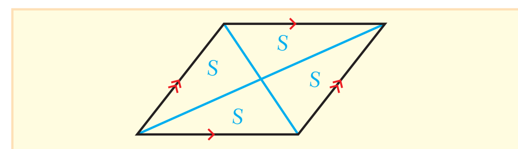




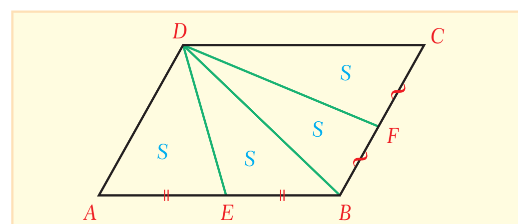
1. Any diagonal in a parallelogram divides it into two equal parts.



2. The two diagonals of a parallelogram divide its area into four equal parts.

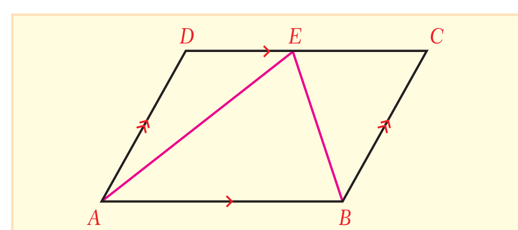


3. Three lines drawn from any vertex of a parallelogram to the opposite vertex and the midpoints of the two opposite sides divide the parallelogram into four equal parts.

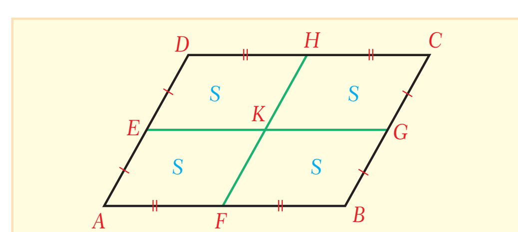


4. Any point E on any side of a parallelogram which is connected to the two non-adjacent vertices creates a triangle which has half area of the parallelogram: in the figure,

$$A(\Delta ABE) = \frac{A(ABCD)}{2}.$$



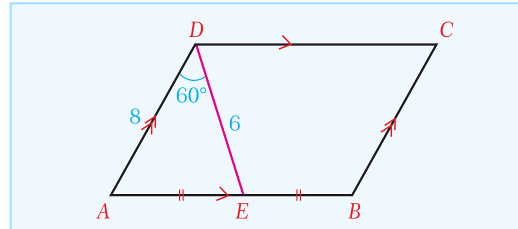
5. Connecting the midpoints of opposite sides of a parallelogram creates four congruent parallelograms.



EXAMPLE 61

In the figure, $ABCD$ is a parallelogram and E is the midpoint of side AB.

Given that $DE = 6$ cm, $AD = 8$ cm and $m(\angle ADE) = 60°$, find the area of $ABCD$.



Solution By the trigonometric formula for the area of a triangle,

$$A(\Delta ADE) = \frac{1}{2} \cdot 6 \cdot 8 \cdot \sin 60° = \frac{1}{2} \cdot 6 \cdot 8 \cdot \frac{\sqrt{3}}{2} = 12\sqrt{3} \text{ cm}^2.$$

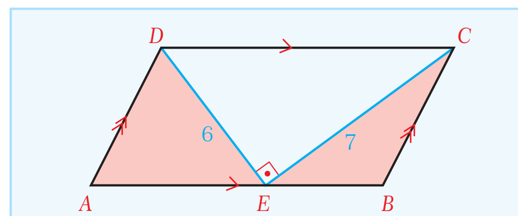
By Property 10.3 we can now write

$A(ABCD) = 4 \cdot A(\Delta ADE) = 4 \cdot 12\sqrt{3} = 48\sqrt{3}$ cm^2.

EXAMPLE 62

In the figure, $ABCD$ is a parallelogram and E is a point on side AB.

Given that $DE = 6$ cm, $CE = 7$ cm and $m(\angle DEC) = 90°$, find the sum of the areas of the shaded regions.

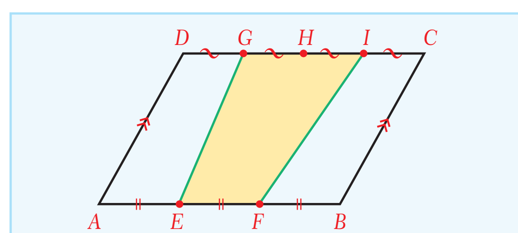


Solution By Property 10.4, $A(\Delta DEC) = \dfrac{A(ABCD)}{2}$ and so the sum of the areas of the shaded regions is also $\dfrac{A(ABCD)}{2}$.

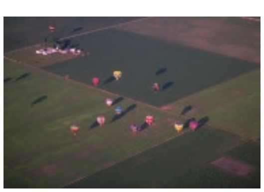
So the sum of the areas of the shaded regions is $\dfrac{A(ABCD)}{2} = A(\Delta DEC) = \dfrac{6 \cdot 7}{2} = 21 \text{ cm}^2$.

EXAMPLE 63

In the figure, $ABCD$ is a parallelogram. Given that $AE = EF = FB$, $DG = GH = HI = IC$ and $A(ABCD) = 120$ cm^2, find the area of quadrilateral $EFIG$.



Solution Let us connect point G to A, B and F.



Then we can find $A(EFIG)$ as the sum of $A(\Delta EGF)$ and $A(\Delta GFI)$.

By Property 10.4 we have $A(\Delta AGB) = \dfrac{A(ABCD)}{2} = \dfrac{120}{2} = 60 \text{ cm}^2$.

In ΔAGB, the base AB is divided into three equal parts. So

$$A(\Delta EGF) = \frac{A(\Delta AGB)}{3} = \frac{60}{3} = 20 \text{ cm}^2.$$

Now let us connect point F to D and C.

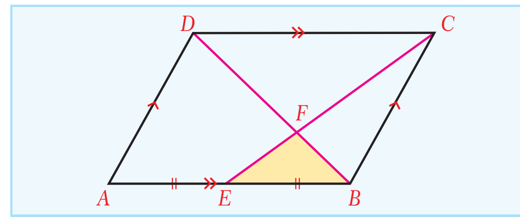
By Property 10.4 we have $A(\Delta DFC) = \dfrac{A(ABCD)}{2} = \dfrac{120}{2} = 60\text{ cm}^2$.

In ΔDFC the base CD is divided into four equal parts and so

$A(\Delta GFI) = 2 \cdot \dfrac{A(\Delta DFC)}{4} = \dfrac{60}{2} = 30\text{ cm}^2$.

So the sum of the shaded areas is $A(\Delta EGF) + A(\Delta GFI) = 20 + 30 = 50\text{ cm}^2$.

EXAMPLE 64 In the figure, $ABCD$ is a parallelogram. E is the midpoint of AB and BD is a diagonal. Given that $A(\Delta BEF) = 8\text{ cm}^2$, find $A(ABCD)$.



Solution If $ABCD$ is a parallelogram then $EB \parallel DC$.

So $m(\angle FBE) = m(\angle FDC)$, $m(\angle FEB) = m(\angle FCD)$ and $m(\angle EFB) = m(\angle DFC)$.

So $\Delta EFB \sim \Delta CFD$ and the ratio of similarity is $k = \dfrac{EB}{DC} = \dfrac{1}{2}$.

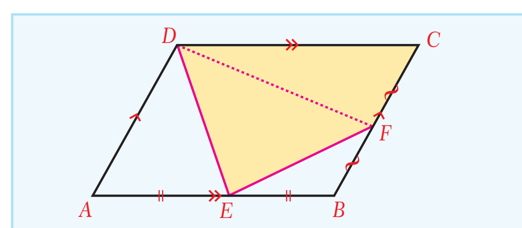
If $k = \dfrac{1}{2}$ then $\dfrac{FB}{DF} = \dfrac{1}{2}$.

Let us draw the segment DE. In ΔDEB, $\dfrac{FB}{DF} = \dfrac{1}{2}$ and $A(\Delta BEF) = 8\text{ cm}^2$.

So $A(\Delta DEF) = 2 \cdot 8 = 16\text{ cm}^2$ and $A(\Delta DEB) = 8 + 16 = 24\text{ cm}^2$.

Finally, by Property 10.3 we can write $A(ABCD) = 4 \cdot A(\Delta DEB) = 4 \cdot 24 = 96\text{ cm}^2$.

EXAMPLE 65 $ABCD$ is a parallelogram and E and F are the midpoints of sides AB and BC respectively. Given that $A(DEFC) = 30\text{ cm}^2$, find $A(ABCD)$.



Solution Let us draw the line segment DF.

By Property 10.3 we can write

$A(\Delta DFC) = \dfrac{A(ABCD)}{4} = A(\Delta ADE)$.

By Property 10.5 we can write

$$A(\Delta EBF) = \dfrac{\dfrac{A(ABCD)}{4}}{2} = \dfrac{A(ABCD)}{8}, \text{ so } A(\Delta DEF) = \dfrac{A(ABCD)}{2} - \dfrac{A(ABCD)}{8} = \dfrac{3}{8} \cdot A(ABCD).$$

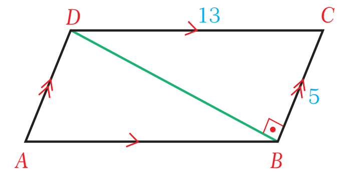
Now $A(DEFC) = A(\Delta DEF) + A(\Delta DFC) = \dfrac{3 \cdot A(ABCD)}{8} + \dfrac{A(ABCD)}{4} = \dfrac{5 \cdot A(ABCD)}{8} = 30\text{ cm}^2$,

which gives $A(ABCD) = \dfrac{30 \cdot 8}{5} = 48\text{ cm}^2$.

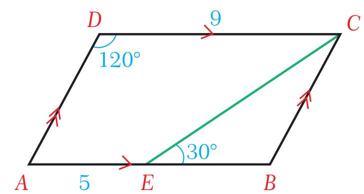
Check Yourself

1. Two sides of a parallelogram measure 14 cm and 18 cm. Given that the acute angles in this parallelogram measure 45°, find the area of the parallelogram.

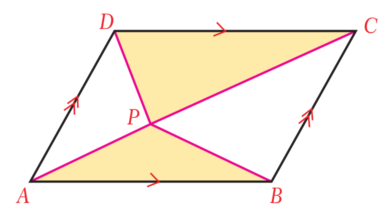
2. In the figure, $ABCD$ is a parallelogram and $m(\angle DBC) = 90°$. Given that $BC = 5$ cm and $DC = 13$ cm, find $A(ABCD)$.



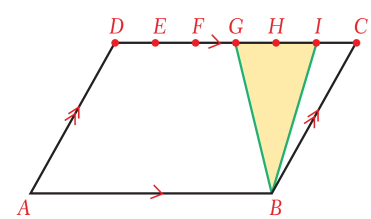
3. In the figure, $ABCD$ is a parallelogram. Given that $m(\angle D) = 120°$, $m(\angle BEC) = 30°$, $AE = 5$ cm and $DC = 9$ cm, find $A(ABCD)$.



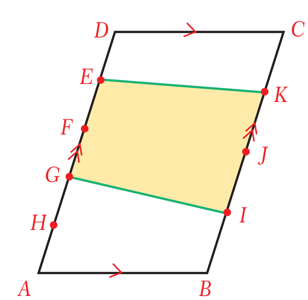
4. In the figure, $ABCD$ is a parallelogram and P is a point inside it. The area of the parallelogram is 76 cm^2. Find the sum of the areas of the shaded regions.



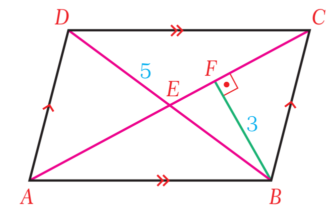
5. In the figure, $ABCD$ is a parallelogram and side DC is divided into six equal parts. Given that $A(ABCD) = 150$ cm^2, find $A(\Delta BIG)$.



6. In the figure, $ABCD$ is a parallelogram. Side AD is divided into five equal parts and side BC is divided into four equal parts. Given that $A(ABCD) = 200$ cm^2, find $A(GIKE)$.



7. In the figure, $ABCD$ is a parallelogram, E is the intersection point of its diagonals, and BF is perpendicular to AC. Given that $BF = 3$ cm, $DE = 5$ cm and $CF = 6$ cm, find the area of the parallelogram $ABCD$.



Answers

1. $126\sqrt{2}$ cm^2 2. 60 cm^2 3. $18\sqrt{3}$ cm^2 4. 38 cm^2 5. 25 cm^2 6. 90 cm^2 7. 60 cm^2

F. AREA OF A RHOMBUS

A rhombus is also a parallelogram, so it shares the rules and properties that we have seen for parallelograms. It also has some additional properties.

Properties 11

A rhombus is a quadrilateral with four congruent sides and parallel opposite sides.

If $ABCD$ is a rhombus then the following statements are true.

1. The area is given by $A(ABCD) = base \times height$:

$$\boxed{A(ABCD) = a \cdot h_a}.$$

2. $A(ABCD) = a^2 \cdot \sin A$
3. Since the diagonals of a rhombus are perpendicular to each other,

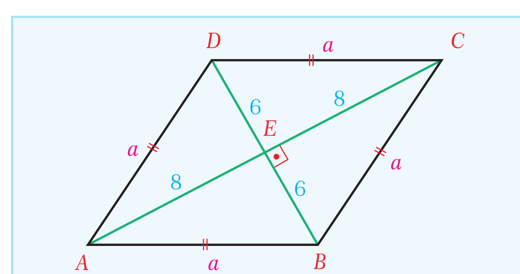
$$A(ABCD) = \frac{AC \cdot BD \cdot \sin 90°}{2} = \frac{AC \cdot BD}{2}.$$

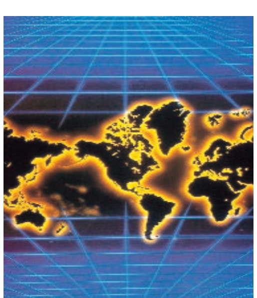
EXAMPLE 66 A rhombus has sides of 6 cm. Given that its height is 4 cm, find the area of this rhombus.

Solution $A(ABCD) = base \times height = 6 \times 4 = 24$ cm^2.

EXAMPLE 67 The diagonals of a rhombus measure 12 cm and 16 cm. Find the length of its altitude.

Solution Let a be the measure of one side of the rhombus. We know that the diagonals of a rhombus bisect each other, so in the figure, $DE = BE = 6$ cm and $AE = EC = 8$ cm.





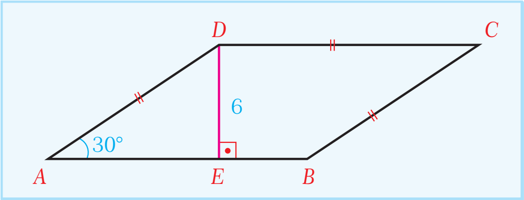
By the Pythagorean Theorem, $6^2 + 8^2 = a^2$, i.e $a = 10$ cm.

Also, $A(ABCD) = \frac{AC \cdot BD}{2} = \frac{16 \cdot 12}{2} = 96$ cm^2.

Since $A(ABCD) = a \cdot h_a$ we have $96 = 10 \cdot h_a$, i.e. $h_a = 9.6$ cm.

EXAMPLE 68 The acute angle of a rhombus measures 30° and its height is 6 cm. Find its area.

Solution Let a be the side length. In the figure, DE is the altitude and $DE = 6$ cm.



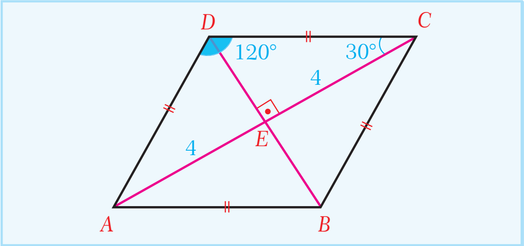
$\sin 30° = \dfrac{DE}{AD}$, so $\dfrac{1}{2} = \dfrac{6}{a}$, i.e.

$AD = a = 12$ cm.

So $A(ABCD) = a \cdot h_a = 12 \cdot 6 = 72$ cm^2.

EXAMPLE 69 The longer diagonal of a rhombus measures 8 cm and the obtuse angle of the rhombus measures 120°. Find the area of this rhombus.

Solution Look at the figure. We know that the diagonals of a rhombus are perpendicular to each other and bisect each other. Also, the diagonals bisect the vertex angles.

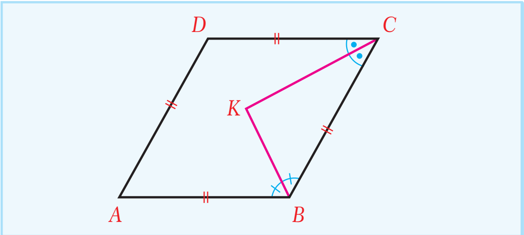


Consider the triangle ΔDEC in the figure. $AC = 8$ cm so $EC = 4$ cm, and by using trigonometric ratios,

$$\tan 30° = \frac{DE}{EC},\ \frac{\sqrt{3}}{3} = \frac{DE}{4},\ DE = \frac{4\sqrt{3}}{3} \text{ cm and } BD = 2 \cdot DE = \frac{8\sqrt{3}}{3} \text{cm}.$$

So by Property 11.3, $A(ABCD) = \dfrac{AC \cdot BD}{2} = \dfrac{8 \cdot \dfrac{8\sqrt{3}}{3}}{2} = \dfrac{32\sqrt{3}}{3}$ cm^2.

EXAMPLE 70 In the figure, $ABCD$ is a rhombus. Given that BK and CK are angle bisectors and $A(\Delta BCK) = 8$ cm^2, find the area of the rhombus.



Solution We know that

$m(\angle ABC) + m(\angle BCD) = 180°$, so

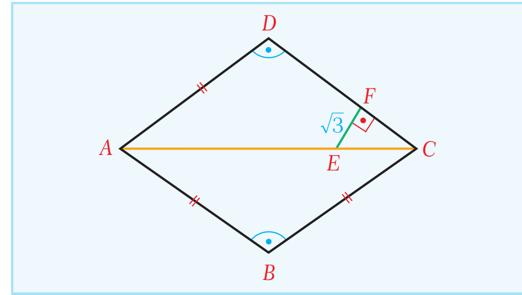
$m(\angle KBC) + m(\angle BCK) = 90°$ and

$m(\angle BKC) = 90°$.

We also know that the diagonals of a rhombus are perpendicular to each other, so K is the intersection point of the diagonals.

By Property 10.2 we can write $A(\Delta BCK) = \dfrac{A(ABCD)}{4}$ and so $A(ABCD) = 4 \cdot 8 = 32$ cm^2.

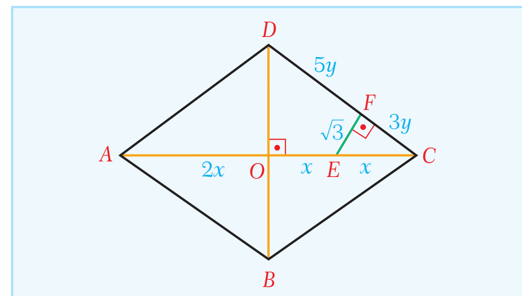
EXAMPLE 71 In the figure, $ABCD$ is a rhombus.
Given that
$AE = 3 \cdot EC$,
$EF \perp DC$,
$3 \cdot DF = 5 \cdot FC$ and $EF = \sqrt{3}$ cm, find the area of the rhombus.



Solution $AE = 3 \cdot EC$ so let $EC = x$ and $AE = 3x$.

$3 \cdot DF = 5 \cdot FC$ so let $DF = 5y$ and $FC = 3y$.

Now let us draw the diagonal BD and mark the intersection point O of the diagonals.



$AC = 4x$ so $OC = 2x$.

$m(\angle EFC) = 90° = m(\angle DOC)$ and

$\angle OCD$ is a common angle, so

$m(\angle FEC) = m(\angle ODC)$ and $\Delta EFC \sim \Delta DOC$.

By similarity, $\dfrac{EF}{DO} = \dfrac{EC}{DC} = \dfrac{FC}{OC}$, i.e $\dfrac{x}{8y} = \dfrac{3y}{2x}$ and so $x^2 = 12y^2$.

By the Pythagorean Theorem in ΔEFC,

$(\sqrt{3})^2 + (3y)^2 = x^2 = 12y^2$, so $y = 1$ and $x = 2\sqrt{3}$ which gives us

$OC = 4\sqrt{3}$ cm, $AC = 8\sqrt{3}$ cm, $DC = 8$ cm.

By the Pythagorean Theorem in ΔDOC,

$DO^2 + OC^2 = DC^2$, so $DO^2 = 8^2 - (4\sqrt{3})^2 = 64 - 48 = 16$, i.e. $DO = 4$ cm and $BD = 8$ cm.

Finally, $A(ABCD) = \dfrac{AC \cdot BD}{2} = \dfrac{8\sqrt{3} \cdot 8}{2} = 32\sqrt{3}$ cm^2.

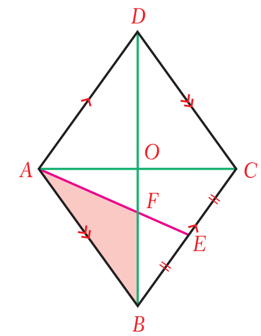
Check Yourself

1. One of the diagonals of a rhombus is the same length as one side of the rhombus. Find the area of the rhombus if one side measures 8 cm.
2. The altitude BH is drawn from vertex B of a rhombus to side CD. Given that $DH = 2$ cm and $CH = 3$ cm, find the area of this rhombus.
3. A rhombus has perimeter 80 cm. Given that one of the diagonals has length 24 cm, find the area of the rhombus.
4. One side of a rhombus measures 18 cm. Given that an obtuse angle in the rhombus measures 150°, find the area of the rhombus.

5. The diagonals of a rhombus are 25 cm and 30 cm long. Find its area.

6. The lengths of the diagonals of a rhombus have ratio 3 : 4. Given that the area of this rhombus is 216 unit^2, find the lengths of the diagonals and one side of the rhombus.

7. In the figure, $ABCD$ is a rhombus and $A(ABCD) = 120$ cm^2. E is the midpoint of BC. Find $A(\Delta ABF)$.



Answers

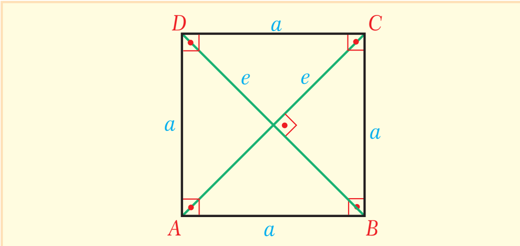
1. $32\sqrt{3}$ cm^2 2. 20 cm^2 3. 384 cm^2 4. 162 cm^2 5. 375 cm^2
6. diagonals: 24, 18; side length: 15 7. 20 cm^2

G. AREA OF A SQUARE

A square is a special type of rectangle, so we can find its area by multiplying the lengths of its two adjacent sides: in the figure,

$$A(ABCD) = a \cdot a, \text{ i.e.}$$

$$\boxed{A(ABCD) = a^2}.$$



The diagonals of a square are perpendicular to each other. Also, if we say that the length of a diagonal is e then the area of the square is

$$\boxed{A(ABCD) = \frac{e^2}{2}}.$$

A square is also a parallelogram, so it shares all the properties of a parallelogram.

EXAMPLE 72 Find the area of a square whose perimeter is 28 cm.

Solution The perimeter is $4 \cdot a = 28$, so $a = 7$ cm. So the area is $a^2 = 7^2 = 49$ cm^2.

EXAMPLE 73 The length of the diagonal of a square is 12 cm. Find the area of this square.

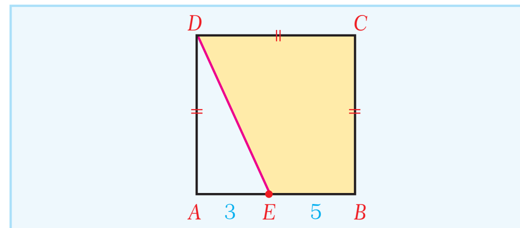
Solution If e is the length of the diagonal then $A_{\text{square}} = \dfrac{e^2}{2}$, i.e. $A = \dfrac{12^2}{2} = \dfrac{144}{2} = 72$ cm^2.

EXAMPLE 74 A square has area 20 cm^2. Find the length of its diagonal.

Solution If the area is a^2 then $20 = a^2$ and so $a = \sqrt{20} = 2\sqrt{5}$ cm.

The length of the diagonal is therefore $e = a\sqrt{2} = 2\sqrt{5} \cdot \sqrt{2} = 2\sqrt{10}$ cm.

EXAMPLE 75 In the figure, $ABCD$ is a square. Given that $AE = 3$ cm and $EB = 5$ cm, find the area of the shaded region.

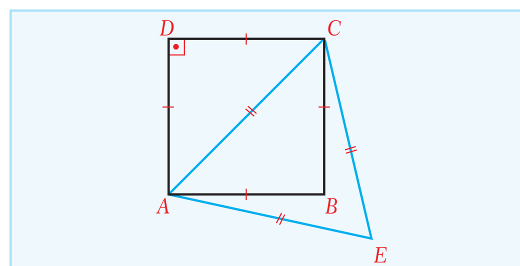


Solution Since $AB = AE + EB$, we have

$a = 3 + 5 = 8$ cm.

So the shaded region has area $A(ABCD) - A(\Delta ADE) = 8^2 - \dfrac{8 \cdot 3}{2} = 64 - 12 = 52$ cm^2.

EXAMPLE 76 In the figure, $ABCD$ is a square and ΔAEC is equilateral triangle. Given that $A(\Delta AEC) = 9\sqrt{3}$ cm^2, find the area of this square.



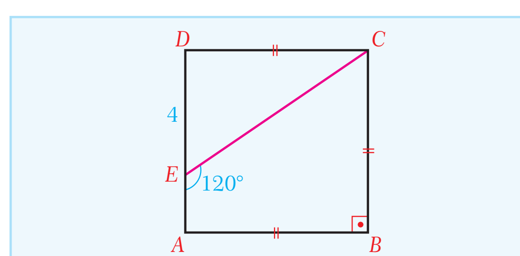
Solution Let one side of the square be a and let the diagonal $AC = e$.

So $A(\Delta AEC) = \dfrac{e^2\sqrt{3}}{4}$, i.e. $9\sqrt{3} = \dfrac{e^2\sqrt{3}}{4}$, $e^2 = 36$. So $e = 6$ cm.

Since $ABCD$ is a square, $A(ABCD) = \dfrac{e^2}{2} = \dfrac{6^2}{2} = \dfrac{36}{2} = 18$ cm^2.

The area A of an equilateral triangle with side length a is $A = \dfrac{a^2\sqrt{3}}{4}$.

EXAMPLE 77 In the figure, $ABCD$ is a square. Given that $DE = 4$ cm and $m(\angle AEC) = 120°$, find the area of the square.



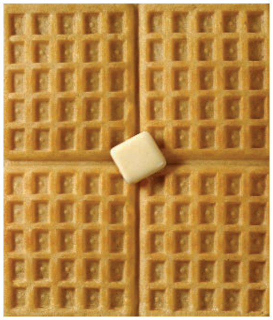
Solution From the figure we can see $m(\angle DEC) = 60°$.

Let one side of the square be a, so $DC = a$.

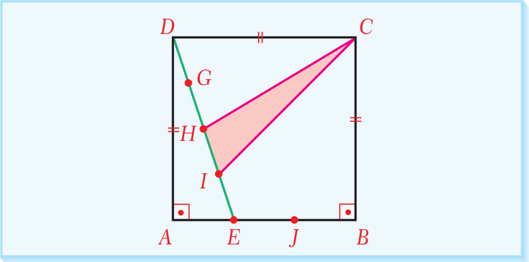
Also, $\tan 60° = \dfrac{DC}{DE}$ which gives us

$\sqrt{3} = \dfrac{a}{4}$, $a = 4\sqrt{3}$ cm.

So $A(ABCD) = a^2 = (4\sqrt{3})^2 = 48$ cm^2.



EXAMPLE 78 In the figure, $ABCD$ is a square. Side AB is divided into three equal parts and side DE is divided into four equal parts. Given that the area of the shaded region is 3 cm^2, find $A(ABCD)$.



Solution Let us draw the line EC.

$A(\Delta CHI) = 3$ is given, so

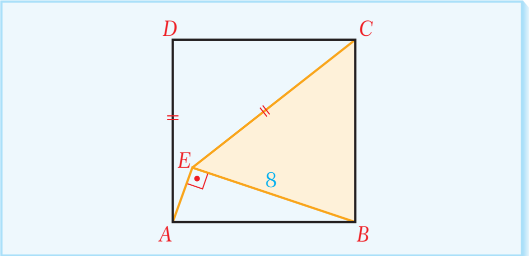
$A(\Delta DEC) = 4 \cdot 3 = 12$ cm^2.

From the properties of a parallelogram we know that $A(\Delta DEC) = \dfrac{A(ABCD)}{2}$.

So $A(ABCD) = 2 \cdot 12 = 24$ cm^2.

EXAMPLE 79 In the figure, $ABCD$ is a square.

Given that $AD = EC$, $EB = 8$ cm and $m(\angle AEB) = 90°$, find $A(\Delta EBC)$.



Solution We know that $EC = AD = BC$, so ΔEBC is an isosceles triangle. Let us draw the altitude CH in ΔEBC.

Since ΔEBC is isosceles, CH is also a median so $EH = HB = 4$ cm.

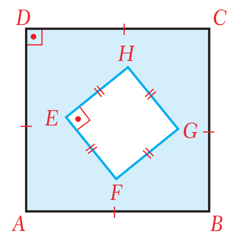
Let $m(\angle HBC) = x$, then $m(\angle EBA) = m(\angle BCH) = 90° - x$ and $m(\angle EAB) = x$.

$m(\angle CHB) = m(\angle AEB) = 90°$ and $AB = BC$, so by the ASA Congruence Theorem we can say $\Delta AEB \cong \Delta BHC$ and $CH = EB = 8$ cm.

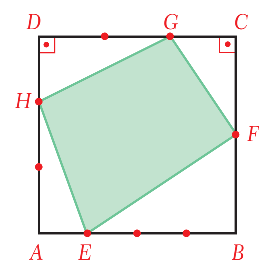
So $A(\Delta EBC) = \dfrac{EB \cdot CH}{2} = \dfrac{8 \cdot 8}{2} = 32$ cm^2.

Check Yourself 12

1. A square has area x cm^2 and perimeter x cm. What is the value of x?
2. The perimeter of a square is 8 cm. Find the area of this square.
3. In the figure, $ABCD$ and $EFGH$ are two squares. Given that sum of the perimeters of these squares is 36 cm and area of the shaded region is 27 cm^2, find the areas of the two squares.



4. The diagonal of a square is 8 cm long. Each side of the square is extended by 1 cm. By how much does the area of square increase?
5. The perimeter of a square $ABCD$ and the perimeter of an equilateral triangle EFG are equal. Find $\dfrac{A(\Delta EFG)}{A(ABCD)}$.
6. In the figure, $ABCD$ is a square with side length 12 cm. The side AB is divided into four equal parts, BC is divided into two equal parts, and DC and AD are each divided into three equal parts. Find the area of the shaded region.



Answers

1. 16 2. 4 cm^2 3. 36 cm^2, 9 cm^2 4. $(1 + 8\sqrt{2})$ cm^2 5. $\dfrac{4\sqrt{3}}{9}$ 6. 77 cm^2

H. AREA OF A TRAPEZOID

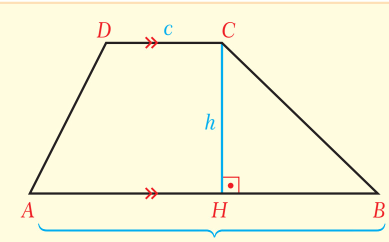
Theorem

A trapezoid is a quadrilateral with two parallel sides.

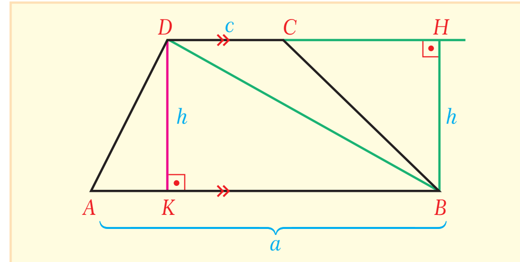
area of a trapezoid

The area of a trapezoid is the product of the height and half the sum of the bases: in the figure,

$$\boxed{A(ABCD) = \frac{a+c}{2} \cdot h}.$$



Proof Look at the figure. Let us draw the altitudes DK and BH.



We know $AB \parallel DC$, so $DK = BH = h$.

Now we draw the diagonal BD.

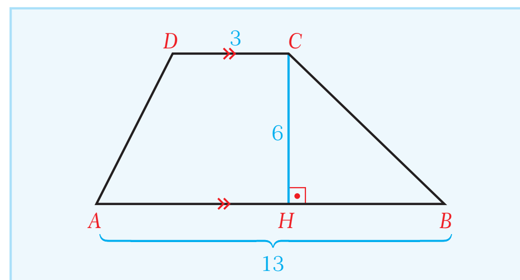
So $A(ABCD) = A(\Delta ABD) + A(\Delta BCD)$

$$= \frac{AB \cdot DK}{2} + \frac{DC \cdot BH}{2}$$

$$= \frac{a \cdot h}{2} + \frac{c \cdot h}{2}$$

$$= \frac{(a+c) \cdot h}{2}, \text{ as required.}$$

EXAMPLE 80 In the figure, $ABCD$ is a trapezoid. $AB = 13$ cm, $DC = 3$ cm and $CH = 6$ cm are given. Find the area of this trapezoid.



Solution We are given $a = 13$, $c = 3$ and $h = 6$.

By the formula for the area of a trapezoid,

$$A(ABCD) = \frac{a+c}{2} \cdot h, \text{ i.e. } A(ABCD) = \frac{13+3}{2} \cdot 6 = 8 \cdot 6 = 48 \text{ cm}^2.$$

Note

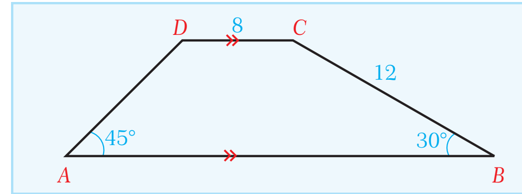
We know that the line which connects the midpoints of the legs of a trapezoid is called a median, and the length of the median is $= \dfrac{a+c}{2}$.

So $A(ABCD) = \dfrac{a+c}{2} \cdot h = \textit{median} \times \textit{height}$.

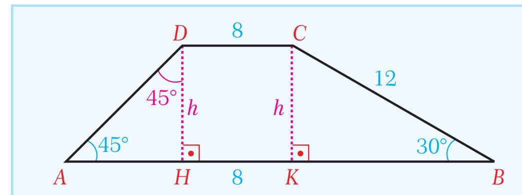
EXAMPLE 81 The median of a trapezoid measures 10 cm and the height is 14 cm. What is the area of this trapezoid?

Solution Area $= \textit{median} \times \textit{height} = 10 \cdot 14 = 140$ cm^2.

EXAMPLE 82 In the figure, $ABCD$ is a trapezoid. Given that $BC = 12$ cm, $DC = 8$ cm, $m(\angle DAB) = 45°$ and $m(\angle ABC) = 30°$, find the area of $ABCD$.



Solution First we draw the altitudes DH and CK.



In ΔCKB, $BC = 12$ cm so $CK = h = 6$ cm and $KB = 6\sqrt{3}$ cm. (30°-60°-90° triangle)

In ΔADH, $m(\angle DAH) = 45°$ and $AH \perp DH$ so $m(\angle ADH) = 45°$.

So $AH = DH$ and we know $DH = CK = 6$ cm.

We also know that $DC = HK = 8$ cm.

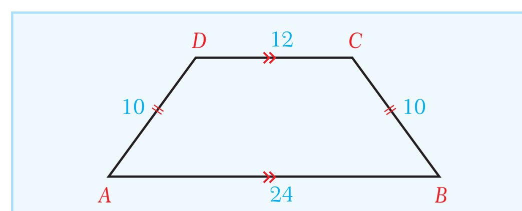
So $a = AH + HK + KB = 6 + 8 + 6\sqrt{3} = (14 + 6\sqrt{3})$ cm, and $c = DC = 8$ cm.

Finally, $A(\Delta ABCD) = \dfrac{a+c}{2} \cdot h = \dfrac{14+6\sqrt{3}+8}{2} \cdot 6 = (22+6\sqrt{3}) \cdot 3 = (66+18\sqrt{3})$ cm^2.

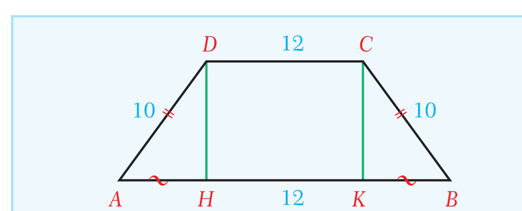
EXAMPLE 83 In the figure, $ABCD$ is an isosceles trapezoid. $AB = 24$ cm, $BC = AD = 10$ cm and $DC = 12$ cm are given.

Find the area of this trapezoid.

An isosceles trapezoid is a trapezoid with congruent legs.



Solution First we draw the altitudes DH and CK to base AB.



So $HK = DC = 12$ cm.

Since $ABCD$ is isosceles, $AH = KB = x$.

But $AB = AH + HK + KB$, which gives

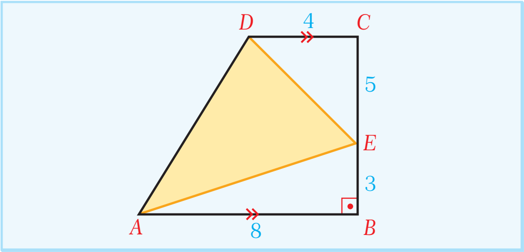
$24 = x + 12 + x$, $x = 6$ cm.

In ΔCKB, $BC = 10$ cm and $KB = 6$ cm. By the Pythagorean Theorem,

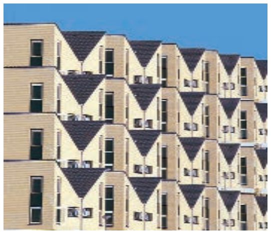
$CK^2 + KB^2 = BC^2$, $h^2 + 6^2 = 10^2$, $h = 8$ cm.

So $A(ABCD) = \dfrac{a+c}{2} \cdot h = \dfrac{24+12}{2} \cdot 8 = 18 \cdot 8 = 144$ cm^2.

EXAMPLE 84 In the figure, $ABCD$ is a right trapezoid. Given that $AB = 8$ cm, $BE = 3$ cm, $EC = 5$ cm and $DC = 4$ cm, find the area of ΔAED.

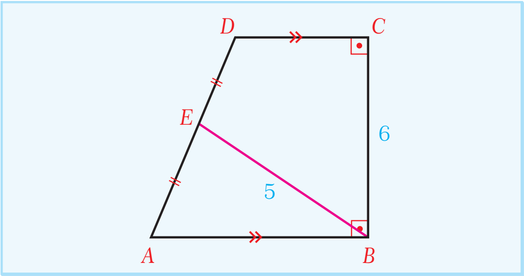


Solution



$$A(\Delta AED) = A(ABCD) - A(\Delta ABE) - A(\Delta DCE)$$
$$= \frac{4+8}{2}\cdot(5+3) - \frac{8\cdot 3}{2} - \frac{4\cdot 5}{2}$$
$$= (6\cdot 8) - 12 - 10$$
$$= 26 \text{ cm}^2.$$

EXAMPLE 85 In the figure, $ABCD$ is a right trapezoid and E is the midpoint of AD. Given that $BC = 6$ cm and $BE = 5$ cm, find the area of the trapezoid.



Solution Let us draw the median EF.

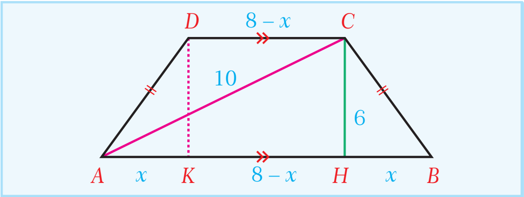
Then $EF \parallel AB \parallel DC$, $EF \perp BC$ and F is the midpoint of BC. So $BF = 3$ cm.

ΔEFB is a right triangle so $EB^2 = EF^2 + FB^2$, i.e. $5^2 = EF^2 + 3^2$, $EF = 4$ cm.

Finally, $A(ABCD) = \textit{median} \cdot \textit{height} = EF \cdot BC = 4 \cdot 6 = 24 \text{ cm}^2$.

EXAMPLE 86 An isosceles trapezoid has a diagonal of 10 cm and height 6 cm. Find the area of this trapezoid.

Solution Let us draw the altitudes DK and CH.



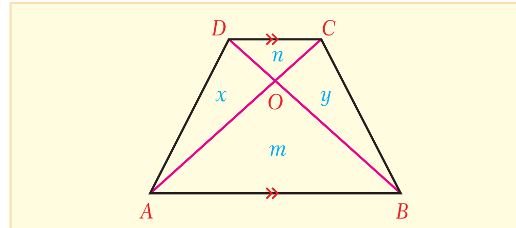
$ABCD$ is an isosceles trapezoid, so let us write $AK = HB = x$.

By the Pythagorean Theorem in ΔAHC we have $AC^2 = AH^2 + CH^2$, i.e. $AH = 8$ cm.

If $AH = 8$ and $AK = x$ then $KH = DC = 8 - x$, $AB = 8 + x$.

So $A(ABCD) = \dfrac{AB + DC}{2}\cdot CH = \dfrac{8 + x + 8 - x}{2}\cdot 6 = 8\cdot 6 = 48 \text{ cm}^2$.

Theorem



In the figure, $ABCD$ is a trapezoid and O is the intersection point of its diagonals. If $A(\Delta AOD) = x$, $A(\Delta AOB) = m$, $A(\Delta BOC) = y$ and $A(\Delta COD) = n$, then

1. $x = y = \sqrt{m \cdot n}$.
2. $A(ABCD) = (\sqrt{m} + \sqrt{n})^2$.

Proof

1. ΔADC and ΔBDC have a common base DC and two common altitudes, so

   $A(\Delta ADC) = A(\Delta BDC)$. So $x + n = y + n$ and therefore $x = y$.

   We know that in any quadrilateral,

   $S_1 \cdot S_3 = S_2 \cdot S_4$.

   This gives us $x \cdot y = m \cdot n$. So $x \cdot y = x^2 = y^2 = m \cdot n$, i.e. $x = y = \sqrt{m \cdot n}$.

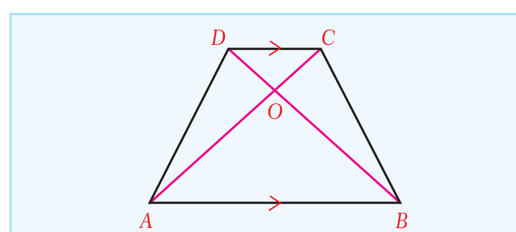
2. $A(ABCD) = x + y + m + n$

$$= \sqrt{m \cdot n} + \sqrt{m \cdot n} + m + n \qquad (x = y = \sqrt{m \cdot n})$$

$$= m + 2 \cdot \sqrt{m \cdot n} + n$$

$$= (\sqrt{m} + \sqrt{n})^2.$$

EXAMPLE 87



In the figure, $ABCD$ is a trapezoid and O is the intersection point of its diagonals. Given that $A(\Delta DOC) = 9\ \text{cm}^2$ and $A(\Delta AOB) = 25\ \text{cm}^2$, find the area of $ABCD$.

Solution

$$A(\Delta AOD) = A(\Delta BOC)$$

$$= \sqrt{A(\Delta DOC) \cdot A(\Delta AOB)}$$

$$= \sqrt{9 \cdot 25}$$

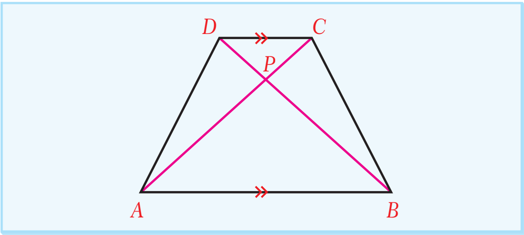
$$= 15\ \text{cm}^2$$

So $A(ABCD) = A(\Delta AOD) + A(BOC) + A(\Delta DOC) + A(AOB)$

$$= 15 + 15 + 9 + 25$$

$$= 64\ \text{cm}^2.$$

EXAMPLE 88 In the figure, $ABCD$ is a trapezoid and P is the intersection point of its diagonals. Given that $A(\Delta ABD) = 12$ cm^2 and $A(\Delta BCD) = 8$ cm^2, find the areas of ΔDPC and ΔPCB.



Solution Let $A(\Delta ADP) = A(\Delta PCB) = x$, then

$A(\Delta DPC) = 8 - x$ and $A(\Delta ABP) = 12 - x$.

By the previous theorem, we have $x = \sqrt{m \cdot n} = \sqrt{(8-x)\cdot(12-x)}$.

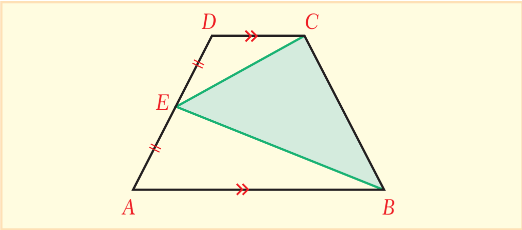
Rearranging this equation given us $x^2 = 96 - 20x + x^2$, i.e. $x = A(\Delta PCB) = \dfrac{24}{5}$ cm^2.

So $A(\Delta DPC) = 8 - x = 8 - \dfrac{24}{5} = \dfrac{16}{5}$ cm^2.

Theorem

Let $ABCD$ be a trapezoid and let E be the midpoint of leg AD.

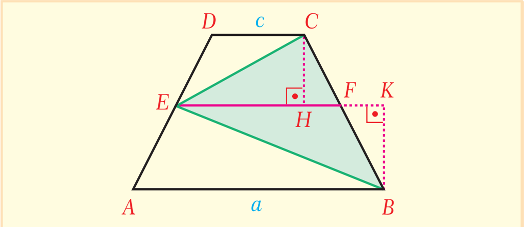
Then $\boxed{A(\Delta BEC) = \dfrac{A(ABCD)}{2}}$.



Proof Let us draw the median EF and the altitudes CH and KB of the trapezoids $DEFC$ and $EABF$ respectively.

Since EF is the median, $CH = BK = \dfrac{h}{2}$ and $EF = \dfrac{a+c}{2}$.

$$\text{Now } A(\Delta ECF) = \frac{EF \cdot CH}{2} = \frac{\frac{a+c}{2} \cdot \frac{h}{2}}{2} = \frac{(a+c)\cdot h}{8}, \text{ and}$$



$$A(\Delta EBF) = \frac{EF \cdot BK}{2} = \frac{\frac{a+c}{2} \cdot \frac{h}{2}}{2} = \frac{(a+c)\cdot h}{8}.$$

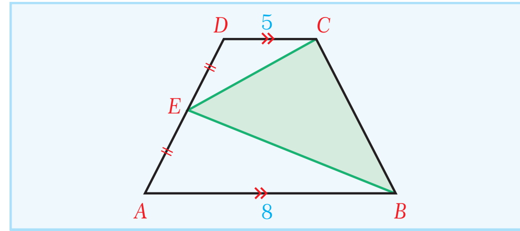
$$\text{So } A(\Delta EBC) = A(\Delta ECF) + A(\Delta EBF) = \frac{(a+c)\cdot h}{8} + \frac{(a+c)\cdot h}{8}$$

$$= \frac{(a+c)\cdot h}{4} = \frac{\frac{a+c}{2} \cdot h}{2}$$

$$= \frac{A(ABCD)}{2}, \text{ as required.}$$

EXAMPLE 89 In the figure, $ABCD$ is a trapezoid and E is the midpoint of leg AD.

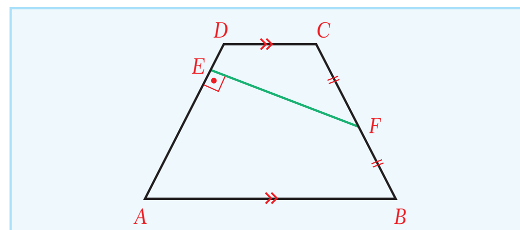
Given that $AB = 8$ cm, $CD = 5$ cm and $A(\Delta BEC) = 26$ cm^2, find the height of the trapezoid.



Solution By the previous theorem we can write $A(ABCD) = 2 \cdot A(\Delta BEC) = 2 \cdot 26 = 52$ cm^2.

But since $A(ABCD) = \dfrac{AB+CD}{2} \cdot h$, we have $52 = \dfrac{8+5}{2} \cdot h$, i.e. $h = 8$ cm.

EXAMPLE 90 In the figure, $ABCD$ is a trapezoid and F is the midpoint of BC.
Given that $AD = 8$ cm,
$EF = 7$ cm and $AD \perp EF$, find the area of the trapezoid.

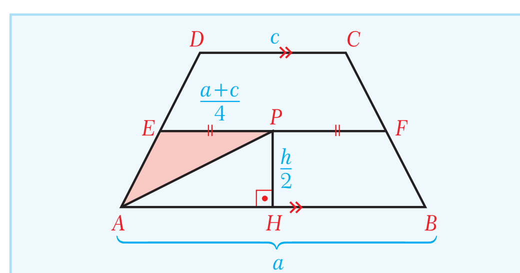


Solution Let us draw DF and AF. By the previous theorem we can write

$$A(ABCD) = 2 \cdot A(\Delta ADF) = 2 \cdot \frac{AD \cdot EF}{2}$$
$$= 8 \cdot 7 = 56 \text{ cm}^2.$$

EXAMPLE 91 $ABCD$ is a trapezoid with bases AB and DC and median EF such that E is on AD and F is on BC. Given that P is the midpoint of EF and $A(\Delta APE) = 4$ cm^2, find $A(ABCD)$.

Solution Let the lengths of the bases be a and c and let the height of the trapezoid be h.



Then $EF = \dfrac{a+c}{2}$,

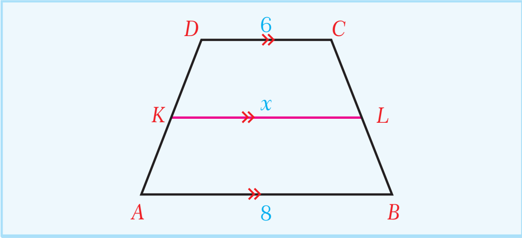
$EP = \dfrac{a+c}{4}$ and $PH = \dfrac{h}{2}$.

$$\text{So } A(\Delta APE) = \frac{EP \cdot PH}{2} = \frac{\frac{a+c}{4} \cdot \frac{h}{2}}{2} = \frac{\frac{a+c}{2} \cdot h}{8}$$
$$= \frac{A(ABCD)}{8} = 4 \text{ cm}^2.$$

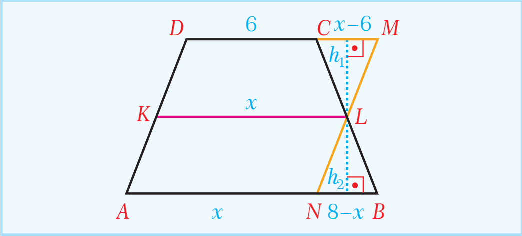
This gives us $A(ABCD) = 8 \cdot 4 = 32$ cm^2.

EXAMPLE 92 In the figure, $ABCD$ is a trapezoid and $KL \parallel AB \parallel DC$.

Given that $A(ABLK) = A(KLCD)$, $AB = 8$ cm and $DC = 6$ cm, find the length $KL = x$.



Solution Let $KL = x$ and let us draw line segment MN parallel to DA, as show in the figure.



So $KL = AN = DM = x$ and $NB = 8 - x$, $CM = x - 6$.

Now let us draw the heights h_1 and h_2 from point L to the bases DM and AB, respectively.

It is given that $A(ABLK) = A(KLCD)$.

So $\frac{8+x}{2} \cdot h_2 = \frac{6+x}{2} \cdot h_1$, i.e. $\frac{h_1}{h_2} = \frac{8+x}{6+x}$. (1)

In ΔCLM and ΔBNL,

$m(\angle CLM) = m(\angle BLN)$, $m(\angle LCM) = m(\angle LBN)$, and

$m(\angle LMC) = m(\angle LNB)$.

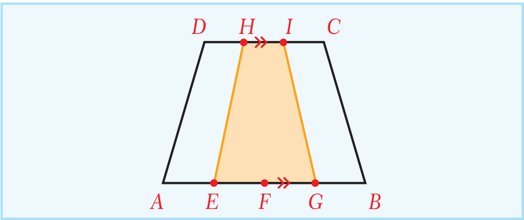
This means $\Delta CLM \sim \Delta BLN$, i.e. $\frac{h_1}{h_2} = \frac{CM}{NB} = \frac{x-6}{8-x}$. (2)

Combining (1) and (2) gives us $\frac{h_1}{h_2} = \frac{8+x}{6+x} = \frac{x-6}{8-x}$, i.e. $64 - x^2 = x^2 - 36$.

Rearranging this expression gives $x^2 = 50$, $x = 5\sqrt{2}$ cm.

EXAMPLE 93 In the figure, $ABCD$ is a trapezoid. Base AB is divided into four equal parts and base DC is divided into three equal parts.

Given that $AB = 2 \cdot DC$, find $\frac{A(EGIH)}{A(ABCD)}$.



Solution Let $DC = x$ so $AB = 2x$, and let h be the height of the trapezoid.

Let us draw the line segments EI, ED and EC.

We can say $A(\Delta DEC) = \frac{xh}{2}$.

$DH = HI = IC$ is given, so $A(\Delta HEI) = \frac{A(\Delta DEC)}{3} = \frac{xh}{6}$.

Now let us draw the line segments IA and IB.

We can write $A(\Delta AIB) = \frac{2xh}{2} = xh$.

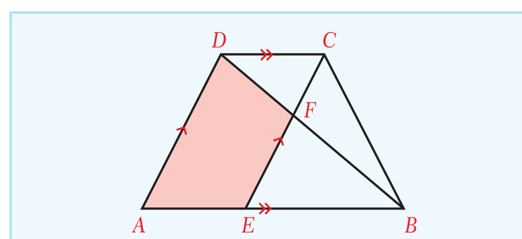
$AE = EF = FG = GB$ is given, so $A(\Delta EIG) = \frac{A(\Delta AIB)}{2} = \frac{xh}{2}$.

So $A(EGIH) = A(\Delta HEI) + A(\Delta EIG) = \frac{xh}{6} + \frac{xh}{2} = \frac{2xh}{3}$.

Finally, $\frac{A(EGIH)}{A(ABCD)} = \frac{\frac{2xh}{3}}{\frac{3xh}{2}} = \frac{4}{9}$.

EXAMPLE 94 In the figure, $ABCD$ is a trapezoid.

Given that $AD \parallel EC$ and $EB = 2 \cdot AE$, find $\frac{A(AEFD)}{A(ABCD)}$.



Solution Let $AE = x$, then $EB = 2 \cdot AE = 2x$, and $DC = x$ because $AECD$ is parallelogram.

Since $DC \parallel EB$, $\Delta DFC \sim \Delta BFE$ and $k = \frac{DC}{EB} = \frac{x}{2x} = \frac{1}{2}$. So $EF = 2 \cdot CF$ and $BF = 2 \cdot DF$.

By Property 9, $\frac{A(\Delta DFC)}{A(\Delta BFE)} = (\frac{1}{2})^2 = \frac{1}{4}$, so if $A(\Delta DFC) = S$ then $A(\Delta BFE) = 4S$.

Property 9: If two triangles are similar then the ratio of their areas is equal to the square of the ratio of similarity.

If $A(\Delta BFE) = 4S$ then $A(\Delta BFC) = 2S$ because $EF = 2 \cdot CF$.

Now let us draw DE. In ΔDEB, $BF = 2 \cdot DF$ and since $A(\Delta BFE) = 4S$, $A(\Delta DEF) = 2S$.

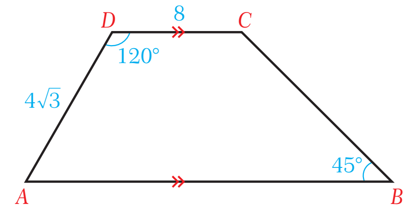
In ΔABD, $EB = 2 \cdot AE$ and $A(\Delta DEB) = 4S + 2S = 6S$, so $A(\Delta ADE) = \frac{A(\Delta DEB)}{2} = 3S$.

So $\frac{A(AEFD)}{A(ABCD)} = \frac{A(\Delta ADE) + A(\Delta DEF)}{A(ABCD)} = \frac{3S + 2S}{12S} = \frac{5}{12}$.

Check Yourself 13

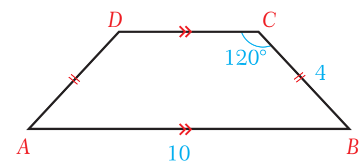
1. The bases of a trapezoid measure 5 cm and 9 cm. Given that the height of this trapezoid is 6 cm, find its area.

2. In the figure, $ABCD$ is a trapezoid. Given that $AD = 4\sqrt{3}$ cm, $DC = 8$ cm, $m(\angle ADC) = 120°$ and $m(\angle ABC) = 45°$, find the area of this trapezoid.

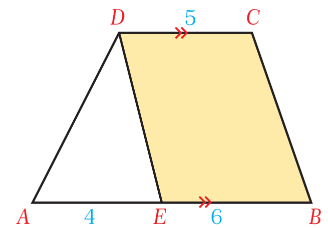


3. The length of the shorter base of a right trapezoid measures 6 cm. The length of the longer base is 12 cm and one of the base angles measures 60°. Find the area of this trapezoid.

4. In the figure, $ABCD$ is an isosceles trapezoid. Given that $AB = 10$ cm, $BC = 4$ cm and $m(\angle BCD) = 120°$, find the area of the trapezoid.

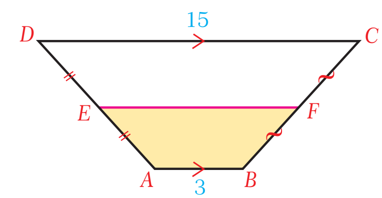


5. In the figure, $ABCD$ is a trapezoid and E is a point on AB. Given that $AE = 4$ cm, $EB = 6$ cm and $DC = 5$ cm, find $\dfrac{A(DEBC)}{A(ABCD)}$.



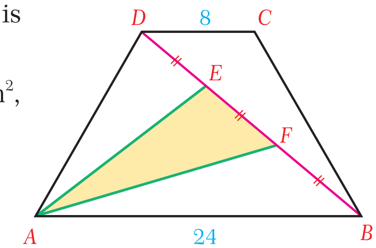
6. The ratio of the lengths of the bases of a trapezoid is 6 : 13. The height of the trapezoid is 20 cm and its area is 380 cm^2. Find the length of the longer base.

7. In the figure, $ABCD$ is a trapezoid and EF is its median. Given that $AB = 3$ cm, $DC = 15$ cm and $A(ABCD) = 90$ cm^2, find $A(ABFE)$.



8. $ABCD$ is a trapezoid and O is the intersection point of its diagonals. Given that $A(\Delta ABC) = 18$ cm^2 and $A(\Delta ACD) = 12$ cm^2, find $A(\Delta AOD)$.

9. In the figure, $ABCD$ is a trapezoid and its diagonal BD is divided into three equal parts.
Given that $AB = 24$ cm, $DC = 8$ cm and $A(\Delta AEF) = 12$ cm^2, find the area of the trapezoid.



Answers

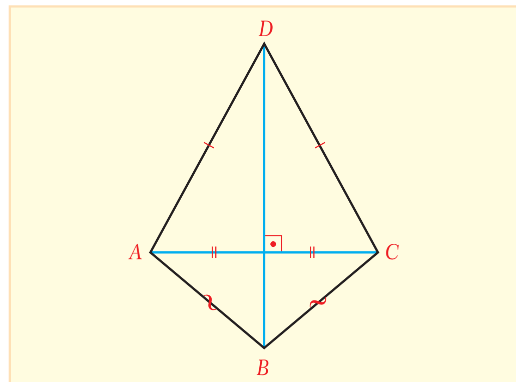
1. 42 cm^2 2. $(66 + 6\sqrt{3})$ cm^2 3. $54\sqrt{3}$ cm^2 4. $16\sqrt{3}$ cm^2 5. $\frac{11}{15}$ 6. 26 cm 7. 30 cm^2
8. $\frac{36}{5}$ cm^2 9. 48 cm^2

I. AREA OF A KITE

If the diagonals of a quadrilateral are perpendicular to each other then the area of the quadrilateral is $A(ABCD) = \frac{AC \cdot BD}{2}$.

Recall that a kite is a quadrilateral that has two pairs of adjacent congruent sides. The diagonals of a kite are perpendicular to each other, so the area of a kite is half the product of its diagonals: in the figure,

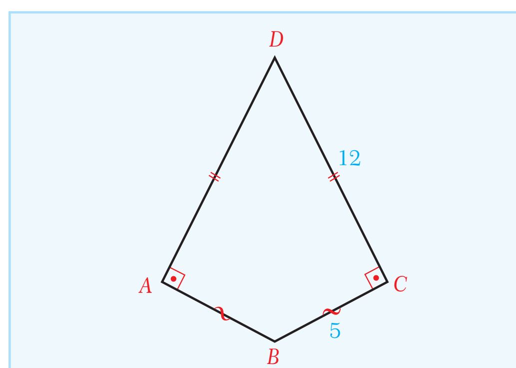
$$\boxed{A(ABCD) = \frac{AC \cdot BD}{2}}.$$



EXAMPLE 95 The diagonals of a kite measure 12 cm and 8 cm. Find the area of this kite.

Solution $A = \frac{AC \cdot BD}{2} = \frac{12 \cdot 8}{2} = 48 \text{ cm}^2$.

EXAMPLE 96 In the figure, $ABCD$ is a kite.
Given that $BC = 5$ cm, $DC = 12$ cm and $m(\angle BCD) = 90°$, find the area of the kite.



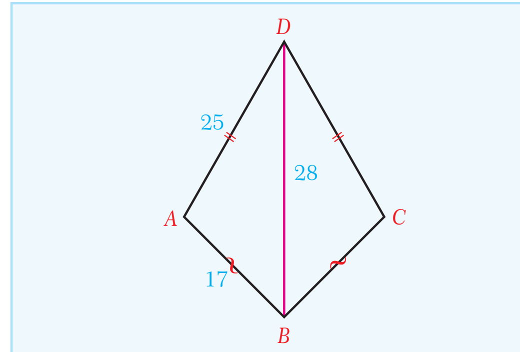
Solution We draw the diagonal BD to get two triangles ΔABD and ΔBCD. Since a kite is symmetric about its main diagonal,
$A(\Delta ABD) = A(\Delta BCD)$ and $m(\angle BAD) = m(\angle BCD) = 90°$.

So $A(ABCD) = 2 \cdot (\Delta BCD) = 2 \cdot \frac{5 \cdot 12}{2} = 60 \text{ cm}^2$.

EXAMPLE 97 In the figure, $ABCD$ is a kite.

Given that $AB = BC = 17$ cm,

$AD = DC = 25$ cm and $BD = 28$ cm, find the area of the kite.



Solution Let us draw the diagonal AC and let the intersection point of the diagonals be O.

Let $AO = y$ and $BO = x$, then

$OD = 28 - x$.

By the Pythagorean Theorem in ΔADO we have

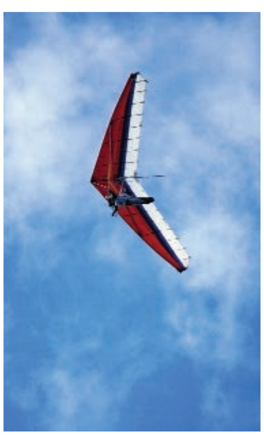
$y^2 = 25^2 - (28 - x)^2 = 17^2 - x^2$, i.e. $625 - 784 + 56x - x^2 = 289 - x^2$.

Rearranging this expression gives us $x = 8$ cm.

Similarly, in ΔABO we have $y^2 = 17^2 - x^2 = 17^2 - 8^2 = 289 - 64 = 225$,

i.e. $y = AO = 15$ cm and $AC = 2 \cdot 15 = 30$ cm.

So $A(ABCD) = \dfrac{AC \cdot BD}{2} = \dfrac{30 \cdot 28}{2} = 210 \text{ cm}^2$.

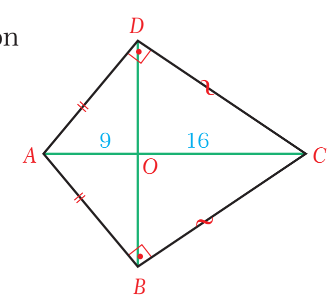


Check Yourself

1. A kite has an area of 120 cm^2. Given that one of its diagonals measures 24 cm, find the length of the other diagonal.

2. In the figure, $ABCD$ is a kite and O is the intersection point of its diagonals.

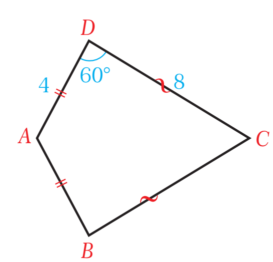
   Given that $AB = AD$, $BC = DC$, $AO = 9$ cm and $OC = 16$ cm, find the area of the kite.



3. In the figure, $ABCD$ is a kite.

   Given that $AB = AD = 4$ cm,

   $BC = DC = 8$ cm and $m(\angle ADC) = 60°$, find the area of the kite.



Answers

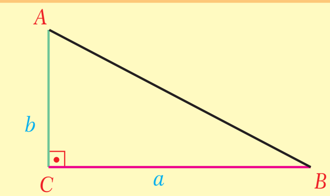
1. 10 cm 2. 300 cm^2 3. $16\sqrt{3}$ cm^2

PROVING THE PYTHAGOREAN THEOREM

The Pythagorean Theorem is one of the oldest and most famous theorems in the history of geometry. Although the theorem was known by the Babylonians and the Egyptians about 1000 years before the time of Pythagoras (who was born in around 575 BC), Pythagoras was the first person to publish a deductive proof, which is why the theorem was given his name.

The Pythagorean Theorem states that the sum of the squares of the legs of a right triangle is equal to the length of its hypotenuse, i.e.

$$\boxed{a^2 + b^2 = c^2}$$



where c is the hypotenuse and a and b are the legs of the right triangle.

Mathematicians since the time of Pythagoras have studied this theorem, and it has been proved in different ways in different branches of mathematics. A writer called Elisha Scott Loomis once published a book with over 360 different proofs of the theorem. Here are four popular proofs.

Proof 1: Let us draw a square $ABCD$ with side length c.

In the figure, ΔABE, ΔBCF, ΔCDG and ΔDAH are congruent right triangles inside $ABCD$ with sides a, b and c.

We can see that the triangles create a smaller square $EFGH$ with side length $a - b$.

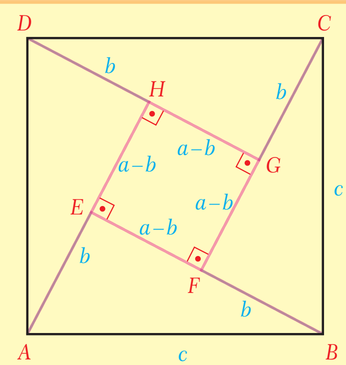
Now we can write the area of $ABCD$ in two ways: $A(ABCD) = c^2$ and

$A(ABCD) = 4 \cdot A(\Delta ABE) + A(EFGH)$.

$$\text{So } c^2 = 4 \cdot \left(\frac{a \cdot b}{2}\right) + (a-b)^2$$

$$= 2ab + a^2 - 2ab + b^2.$$

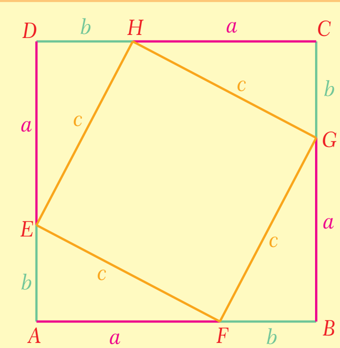
Simplifying this gives $c^2 = a^2 + b^2$.



$AH = DG = CF = BE = a$

Proof 2: Let us draw a square $ABCD$ with side length $a + b$. Then we choose the points E, F, G and H such that ΔEAF, ΔFBG, ΔGCH and ΔHDE are right triangles with sides a, b and c.

We can say that each of these four triangles has hypotenuse c.



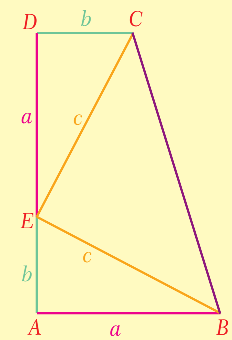
Now we can write $A(ABCD)$ in two different ways:

$A(ABCD) = (a + b)^2$ and $A(ABCD) = 4 \cdot A(\Delta EAF) + A(EFGH)$.

So $A(ABCD) = (a+b)^2 = 4 \cdot (\frac{a \cdot b}{2}) + c^2$, i.e. $a^2 + 2ab + b^2 = 2ab + c^2$.

Canceling $2ab$ from each side gives us $a^2 + b^2 = c^2$.

Proof 3: Let us draw the right trapezoid $ABCD$ with bases a and b and height $a + b$. Then we connect B and C with point E so that ΔCDE and ΔEAB are congruent right triangles with sides a, b and c.



As we can see in the figure, ΔBEC is an isosceles triangle with leg c.

We can write the area of the right trapezoid in two ways:

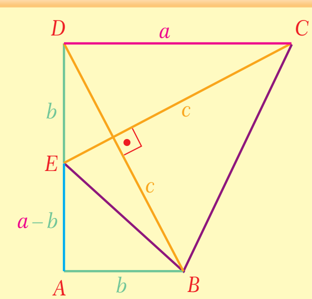
$$A(ABCD) = \frac{\textit{sum of the bases} \cdot \textit{height}}{2},$$

and $A(ABCD) = 2 \cdot A(\Delta EAB) + A(\Delta BEC)$.

So $A(ABCD) = (\frac{a+b}{2}) \cdot (a+b) = 2 \cdot (\frac{a \cdot b}{2}) + \frac{c \cdot c}{2}$, which gives us $\frac{a^2 + 2ab + b^2}{2} = \frac{2ab + c^2}{2}$.

Canceling the denominators and $2ab$ gives us $a^2 + b^2 = c^2$.

Proof 4: Let us draw the right trapezoid $ABCD$ with bases a and b and height a.



We can say that

$EC = DB = c$, $\Delta EDC \cong \Delta BAD$ by the SSS Congruence Theorem.

From the angles we can get $DB \perp EC$.

Now we can write the area of the right trapezoid in two ways:

$$A(ABCD) = \frac{\textit{sum of the bases} \cdot \textit{height}}{2}$$

and $A(ABCD) = A(BCDE) + A(\Delta ABE)$.

So $A(ABCD) = \frac{a+b}{2} \cdot a = \frac{c \cdot c}{2} + \frac{(a-b) \cdot b}{2}$.

Canceling 2 from both sides and taking $ab - b^2$ to the left-hand side gives us $a^2 + b^2 = c^2$.

EXERCISES 3.2

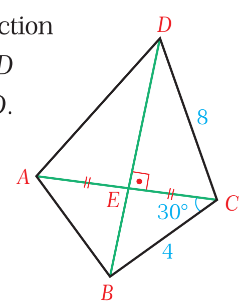
A. Area of a Quadrilateral

1. In the figure,
$DC = 6$,
$AD = 8$,
$AB = 10$,
$BC = 12$ and
$m(\angle ADC) = 90°$.
Find the area of the quadrilateral $ABCD$.

2. $ABCD$ is a convex quadrilateral and E is the intersection point of its diagonals. Given that $AE = 2$, $BE = 5$, $CE = 6$, $DE = 10$ and $BC = 5$, find the area of $ABCD$.

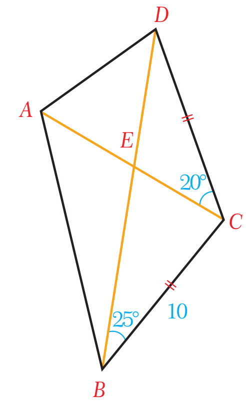
3. $ABCD$ is a convex quadrilateral and E is the intersection point of its diagonals. $DE = 3$ cm and $BE = 12$ cm are given. Find $\dfrac{A(\Delta ADC)}{A(ABCD)}$.

4. In the figure, E is the intersection point of the diagonals of $ABCD$ and AC is perpendicular to BD. Given that $BC = 4$, $DC = 8$, $AE = EC$ and $m(\angle ACB) = 30°$, find the area of $ABCD$.



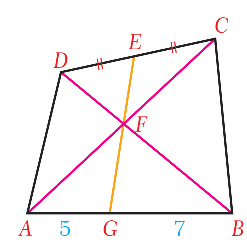
5. $ABCD$ is a convex quadrilateral with $AD = DC = 8$, $BD = 14$ and $m(\angle ADC) = 60°$. The angle between its diagonals is $90°$. What is $A(ABCD)$?

6. In the figure, $ABCD$ is a convex quadrilateral and E is the intersection point of its diagonals.
Given that $DC = BC$,
$m(\angle DCA) = 20°$,
$m(\angle DBC) = 25°$,
$AC = 6$ and
$BD = 10$,
find the area of $ABCD$.



7. $ABCD$ is a convex quadrilateral and E is the intersection point of its diagonals. Given that $A(\Delta ABE) = A(\Delta CDE) = 18$ and $A(\Delta BCE) = 4 \cdot A(\Delta ADE)$, find the area of $ABCD$.

8. ✪ In the figure, $ABCD$ is a convex quadrilateral, E is the midpoint of DC and F is the intersection point of the diagonals. Given that $AG = 5$ cm and $GB = 7$ cm, find $\dfrac{A(\Delta BCF)}{A(\Delta DAF)}$.



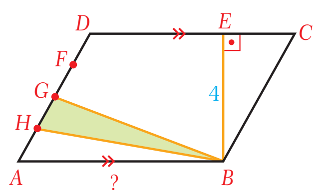
B. Area of a Parallelogram

9. The sides of a parallelogram measure 10 cm and 24 cm. Given that the length of the altitude to the longer side is 5 cm, find the length of the altitude to the shorter side.

10. In the figure, $ABCD$ is a parallelogram, DE is perpendicular to AB and AF is perpendicular to BC. Given that $AB = 12$, $AD = 6$ and
$AF = 8$, find the length $DE = x$.

11. One of the diagonals of a parallelogram has the same length as one of its sides. Given that the longer side of the parallelogram is 6 units long and its interior acute angle measures 30°, find the area of the parallelogram.

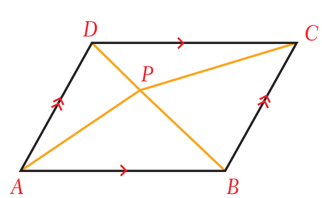
12. In the figure, $ABCD$ is a parallelogram and BE is perpendicular to DC. Given that $AH = HG = GF = FD$, $A(\Delta GBH) = 6$ and $EB = 4$, find the length of AB.



13. $ABCD$ is a parallelogram and H is a point on DC such that $BH \perp DC$. Given that $BH = 6$, $BC = 10$ and $DH = 8$, find the area of $ABCD$.

14. $ABCD$ is a parallelogram and DB is its diagonal. Given that $m(\angle DAB) = 30°$, $AD \perp BD$ and $BC = 8$, find $A(ABCD)$.

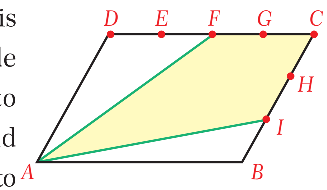
15. In the figure, $ABCD$ is a parallelogram and P is a point inside the parallelogram.



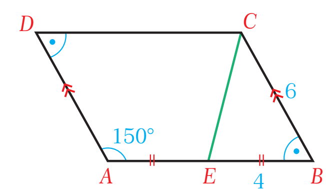
Given that
$A(\Delta PBC) = 18$, $A(\Delta PAD) = 12$ and
$2 \cdot A(\Delta PAB) = 3 \cdot A(\Delta PCD)$, find the area of ΔPAB.

16. One of the diagonals of a parallelogram is 8 units long and one of its sides is 10 units long. The angle between this side and this diagonal is 45°. Find the area of the parallelogram.

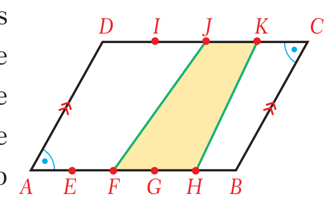
17. In the figure, $ABCD$ is a parallelogram. Side DC is divided into four equal parts and side BC is divided into three equal parts. Given that $A(ABCD) = 240$, find the area of the quadrilateral $AICF$.



18. In the figure, $ABCD$ is a parallelogram and E is the mid-point of side AB. Given that $EB = 4$, $BC = 6$ and $m(\angle DAB) = 150°$, find $A(AECD)$.



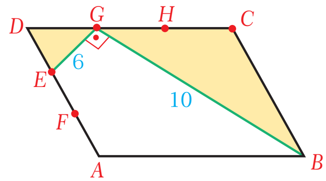
19. In the figure, $ABCD$ is a parallelogram. Side AB is divided into five equal parts and side CD is divided into four equal parts. Given that $A(ABCD) = 180$, find $A(FHKJ)$.



20. $ABCD$ is a parallelogram. E and F are two points on base AB, and G is a point on CD. Given that $EF = \dfrac{3}{5} \cdot AB$ and $A(\Delta EFG) = 12$, find $A(ABCD)$.

21. ✪ $ABCD$ is a parallelogram and E and F are the midpoints of sides AB and BC respectively. If G is the intersection point of lines AF and CE and $A(\Delta AEG) = 12$, find $A(ABCD)$.

22. ✪ In the figure, $ABCD$ is a parallelogram and sides AD and CD are each divided into three equal parts.



Give that $EG \perp GB$, $EG = 6$ and $GB = 10$, find the sum of the areas of the shaded regions.

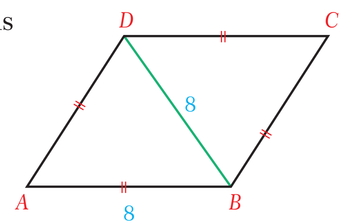
C. Area of a Rhombus

23. A rhombus has area 80 and a diagonal which is 20 units long. Find the length of the other diagonal.

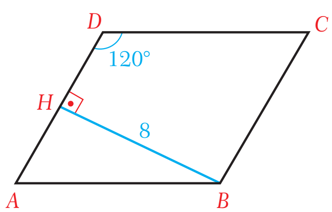
24. A rhombus has diagonals which are 10 cm and 22 cm long. Find its area.

25. In the figure, $ABCD$ is a rhombus.

Given that $AB = BD = 8$, find $A(ABCD)$.



26. In the figure, $ABCD$ is a rhombus and BH is an altitude. Given that $BH = 8$ and $m(\angle ADC) = 120°$, find the area of this rhombus.



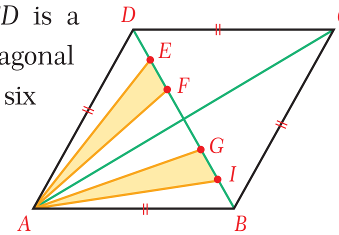
27. A rhombus has area 200 cm^2. Given that the length of the altitude to a side is 10 cm, find the measure of the acute angle of this rhombus.

28. In a rhombus $ABCD$, the lengths of the diagonals have the ratio $5\sqrt{2}$. Given that $A(ABCD) = 120\sqrt{2}$, find the lengths of the diagonals.

29. The sum of the lengths of the diagonals of a rhombus is 34 and one side of the rhombus measures 13 units. Find the area of this rhombus.

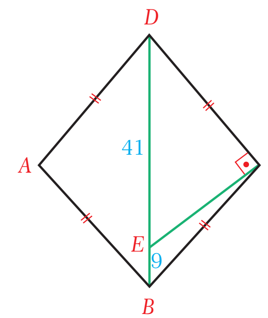
30. In the figure, $ABCD$ is a rhombus and the diagonal BD is divided into six equal lengths.

Given that $A(ABCD) = 72$,

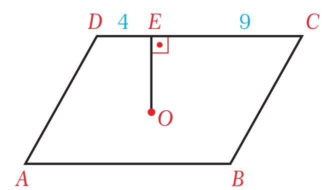


find the sum of the areas of the shaded regions.

31. In the figure, $ABCD$ is a rhombus. Point E is on the diagonal BD such that $DE = 41$, $EB = 9$ and $m(\angle DCE) = 90°$. Find the area of this rhombus.

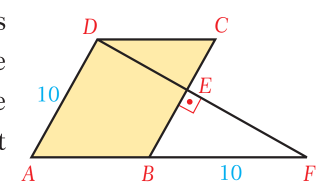


32. In the figure, $ABCD$ is a rhombus, O is the intersection point of its diagonals, E is a point on DC, and



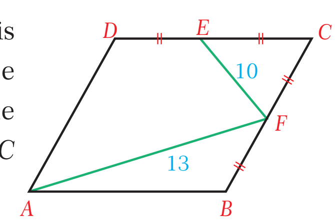
$OE \perp DC$. Given that $DE = 4$, $EC = 9$ and $m(\angle OEC) = 90°$, find the area of the rhombus.

33. In the figure, $ABCD$ is a rhombus and the points A, B and F are collinear. Given that $DF \perp BC$ and



$AD = BF = 10$, find the area of rhombus $ABCD$.

34. ✪ In the figure, $ABCD$ is a rhombus and the points E and F are the midpoints of sides DC and BC, respectively. Given that $EF = 10$ and $AF = 13$, find $A(ABCD)$.

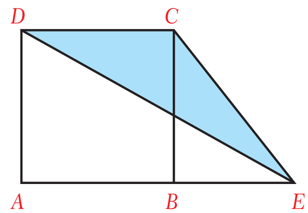


D. Area of a Square

35. Find the area of the square whose perimeter is $12\sqrt{5}$ units.

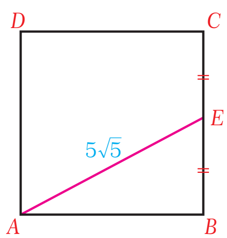
36. The diagonal of a square is 6 units long. Find the area of this square.

37. In the figure, $ABCD$ is a square and A, B and E are collinear. Given that $A(\Delta EDC) = 20$, find the area of the square.



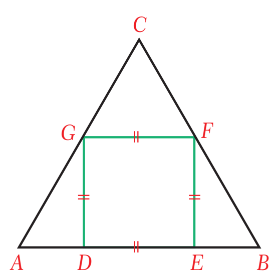
38. A rectangular floor has side lengths 12 m and 15 m. We want to cover it with square tiles with side length 40 cm. How many tiles do we need?

39. In the figure, $ABCD$ is a square and E is the midpoint of BC. Given that $AE = 5\sqrt{5}$, find the area of this square.



40. If we lengthen the sides of a square by 40%, by what percentage will its area increase?

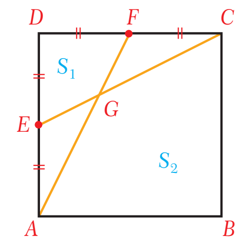
41. In the figure, ΔABC is an equilateral triangle and $DEFG$ is a square. Given that $A(\Delta ABC) = 4\sqrt{3}$, find $A(DEFG)$.



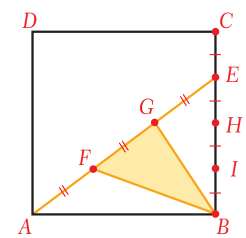
42. Two opposite sides of a square are shortened by 2 units. The area of the rectangle obtained is 35 square units. What was the length of one side of the original square?

43. $ABCD$ and $BCEF$ are two squares, and P and Q are the respective intersection points of their diagonals. Given that $AB = 8$, find the area of $PBQC$.

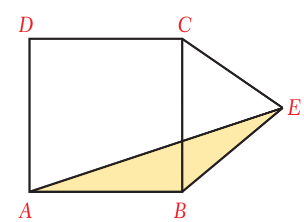
44. In the figure, $ABCD$ is a square and E and F are the midpoints of sides AD and DC respectively. Given that $A(DEGF) = S_1$ and $A(ABCG) = S_2$, find $\dfrac{S_1}{S_2}$.



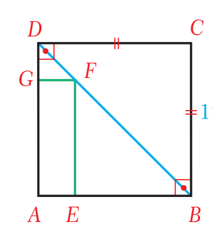
45. In the figure, $ABCD$ is a square. Side BC is divided into four equal parts and AE is divided into three equal parts. Find the area of the shaded region if $A(ABCD) = 120$.



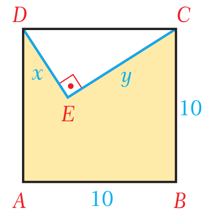
46. In the figure, $ABCD$ is a square and ΔBEC is an equilateral triangle. Given that $A(\Delta ABE) = 16$, find the area of the square.



47. In the figure, $ABCD$ is a square with side length 12 cm and $AEFG$ is a rectangle. Given that $BF = 3 \cdot DF$, find $\dfrac{A(AEFG)}{A(ABCD)}$.



48. In the figure, $ABCD$ is a square with side length 10 units. Given that $DE + EC = 12$ and $m(\angle DEC) = 90°$, find the area of the shaded region $ABCED$.

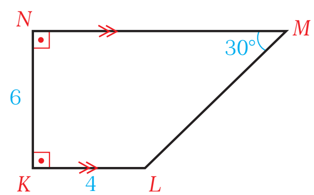


E. Area of a Trapezoid

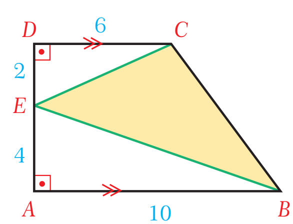
49. The two bases of a trapezoid are 12 and 8 units long. Given that the area of the trapezoid is 100 square units, find the length of its altitude.

50. In the figure, $KLMN$ is a right trapezoid.

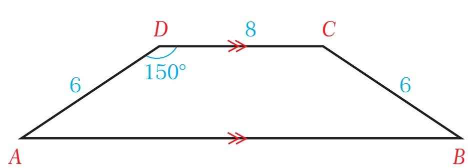
Given that $KL = 4$, $NK = 6$ and $m(\angle LMN) = 30°$, find the area of the trapezoid.



51. In the figure, $ABCD$ is a right trapezoid and E is a point on AD. Given that $AB = 10$, $DC = 6$, $AE = 4$ and $ED = 2$, find the area of ΔEBC.

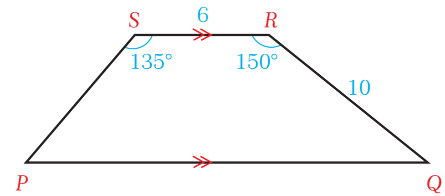


52.



In the figure, $ABCD$ is an isosceles trapezoid. Given that $AD = BC = 6$, $DC = 8$ and $m(\angle ADC) = 150°$, find the area of the trapezoid.

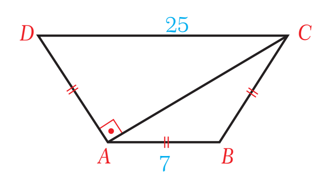
53.



In the figure, $PQRS$ is a trapezoid. Given that $QR = 10$, $SR = 6$, $m(\angle PSR) = 135°$ and $m(\angle SRQ) = 150°$, find the area of the trapezoid $PQRS$.

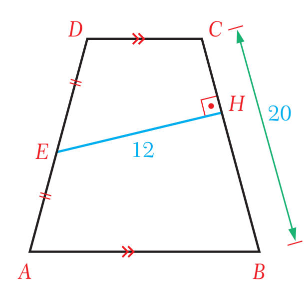
54. MN and PQ are the bases of a trapezoid and O is the intersection point of its diagonals. Given that $A(\Delta MON) = 90$, $MN = 12$ and $PQ = 8$, find the area of trapezoid $MNPQ$.

55. In the figure, $ABCD$ is an isosceles trapezoid. Given that $AD \perp AC$, $AD = AB = BC$, $AB = 7$ and $DC = 25$, find the area of this trapezoid.



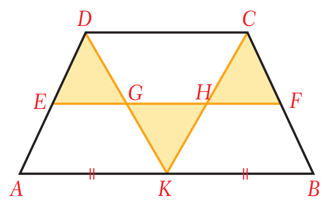
56. The diagonals of an isosceles trapezoid are perpendicular to each other. Given that one diagonal is 20 units long, find the area of this trapezoid.

57. In the figure, $ABCD$ is a trapezoid and E is the midpoint of AD. Given that $EH \perp BC$, $EH = 12$ and $BC = 20$, find the area of $ABCD$.

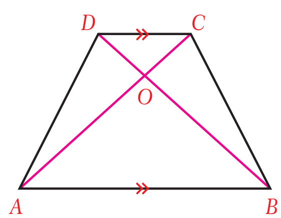


58. AB and DC are the bases of a trapezoid and E is a point on base AB. Given that $AB = 3 \cdot DC$ and $A(\Delta DEC) = 12$, find the area of the trapezoid.

59. In the figure, $ABCD$ is a trapezoid and E, F and K are the midpoints of sides AD, BC and AB, respectively. Given that ΔEDG, ΔHFC and ΔGHK are equilateral triangles and $A(\Delta GHK) = 4$, find the total area of the trapezoid.

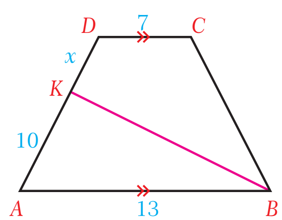


60. In the figure, $ABCD$ is a trapezoid and O is the intersection point of its diagonals. Given that $A(\Delta AOD) = 12$ and $A(\Delta DOC) = 4$, find $A(ABCD)$.



61. In the figure, $ABCD$ is a trapezoid and K is a point on AD.

Given that $AB = 13$, $DC = 7$, $AK = 10$ and $A(\Delta ABK) = A(KBCD)$, find the length $KD = x$.



62. The two bases of a trapezoid measure 5 cm and 15 cm. Given that the diagonals measure 12 cm and 16 cm, find the area of this trapezoid.

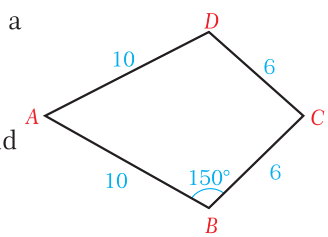
63. ✪ $ABCD$ is a trapezoid with bases AB and DC. $AB = 26$, $BC = 16$, $CD = 6$ and $AD = 12$ are given. Find the area of the trapezoid.

64. ✪ $ABCD$ is an isosceles trapezoid with diagonals 12 units long. Given that the angle between a diagonal and a base is 30°, find the area of the trapezoid.

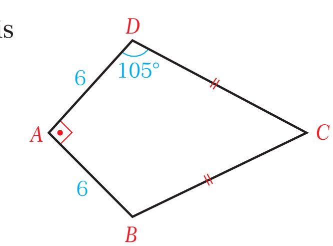
F. Area of a Kite

65. The diagonals of a kite are 6 and 8 units long. Find the area of this kite.

66. In the figure, $ABCD$ is a kite, $AD = AB = 10$, $DC = BC = 6$ and $m(\angle ABC) = 150°$. Find the area of the kite.



67. In the figure, $ABCD$ is a kite. Given that $AB = AD = 6$, $BC = DC$, $m(\angle BAD) = 90°$ and $m(\angle ADC) = 105°$, find the area of the kite.



68. $ABCD$ is a kite. Given that $AB = BC$, $AD = DC = 12$, $m(\angle ADC) = 60°$ and $m(\angle ABC) = 120°$, find the area of the kite.